Network formation among rivals*

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February 13, 2015

Abstract

We study the formation of bilateral agreements among rivals. All else equal, the payoff of an agent increases in his own number of partners and decreases in the number of partners of his rivals. We assume that agents are farisghted: they anticipate that their choice of partners may trigger reactions from their rivals. When more cooperation among equals is profitable, and when the payoff of agents in a small clique increases in the size of the clique, a von-Neumann-Morgenstern farsighted stable set exists. The set contains either two-clique networks, or dominant group networks in which only connected agents are active competitors. Network formation may thus endogenously create a barrier to entry. If the sum of payoffs increases when the connections are more unequally distributed among rivals, the efficient networks are either nested split graphs, or have a core-periphery structure. The networks formed by farsighted rivals are not efficient. We show that standard economic models of network formation among rivals satisfy the above properties.

JEL classification: C70, D20, D40.

Keywords: Network formation, Competition, Rivalry, Farsightedness, Efficiency.

1 Introduction

Examples of cooperation between rivals are abundant. Firms who are competitors on a final market jointly invest in R&D to share its costs and benefits, they share customer databases or engage in cross-licensing agreements. Countries sign bilateral trade agreements, colleagues competing for a promotion work in teams, etc. In this paper, we propose a general class of games to analyze these environments. In a *game of network formation among rivals*, ex ante symmetric agents first engage in bilateral cooperation and then compete. The payoff of an agent increases in his own number of partners (degree monotonicity) and decreases in rivals' number of partners

*Gilles Grandjean and Wouter Vergote acknowledge financial support from the FNRS.

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(negative externalities).[1] In this setup, we analyze the networks formed by farsighted agents and contrast these to the efficient networks, i.e. those leading to the highest sum of payoffs.

Farsighted agents forecast how other agents would react to their choice of partners, and take a decision by comparing the current network to the end network that forms when other agents have further deviated. Farsightedness in network formation has received increasing attention over the past few years.[2] In his survey on network formation, Jackson (2005) has stated that:

"...in large networks it might be that players have very little ability to forecast how the network might change in reaction to the addition or deletion of a link. In such situations the myopic solutions are quite reasonable. However, if players have very good information about how others might react to changes in the network, then these are things that one wants to allow for either in the specification of the game or in the definition of the stability concept".

We believe that farsightedness is an appropriate assumption when studying cooperation between competitors, as the number of competitors is usually rather small and the stakes are high. Rivals then have the opportunity and the incentives to foresee how others might react to changes in the network. We capture this by the notion of indirect dominance (Harsanyi 1974). A final network indirectly dominates an initial network if there exists a sequence of networks that implements the final network from the initial network such that at any step of the sequence all agents who deviate have a higher payoff in the final network than in the current network. We use the stable set (von Neumann Morgenstern, 1944), based on indirect dominance as a solution concept. The farsighted stable set is both internally stable - no network of the set indirectly dominates another network of the set - and externally stable - every network outside the set is indirectly dominated by a network belonging to the set. The farsighted stable set can then be interpreted as a standard of behavior when agents are farsighted.

We show that there always exists a farsighted stable set in a game of network formation among rivals satisfying *strong degree monotonicity* and *minority economies to scale*. It is either composed of dominant group networks, or of 2-clique networks. In a dominant group network, each member of the group is connected to the other group members while the remaining agents are not connected and do not take part in the competition. Networking then endogenously creates a barrier to entry. In a network composed of two asymmetric cliques, each agent belongs either to a large or to a small group of connected agents, and each agent is an active competitor. The first property needed to establish this result, strong degree monotonicity, implies that agents who have the same degree find it worthwhile to see their degree increase. The second, minority economies to scale, imposes that the payoff of agents in a small clique increases in the size of

[1]See for instance Goyal and Moraga (2001), Goyal and Joshi (2003), Goyal and Joshi (2006a), Goyal and Joshi (2006b), Marinucci and Vergote (2011), Grandjean et al. (2013) for models of competition in networks competition lying in this class of games.

[2]Approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon, and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Page, Wooders and Kamat (2005), Dutta, Ghosal, and Ray (2005), Herings, Mauleon, and Vannetelbosch (2004), Page and Wooders (2009), Herings, Mauleon, and Vannetelbosch (2014), and Ray and Vohra (2014).

the clique when they are facing another clique with the majority of agents.

We then analyze the efficient networks of a game of network formation among rivals when two properties hold. The first property, *welfare improving switch*, imposes that the sum of payoff increases after a switch - by which the degree of one agent increases while that of a less connected agent decreases - when the agents whose degree decreases remains active in the competition. The second, *switch externality*, imposes that agents not involved in the switch are not hurt by it. We then show that the networks that maximize the sum of payoffs are nested split graphs when agents are active in every network.[3] Otherwise, when lowly connected agents may decide not to participate to the competition, a switch may no longer be welfare improving if it leads to the exclusion of the agent hurt by the switch. We then find that the efficient network is either a core-periphery network or a (quasi-)nested split graph. they are either (quasi-)nested split graphs or core-periphery networks.[4]

The four properties we impose are satisfied in many models of network formation among rivals. We show that this is the case in Goyal and Joshi (2003)'s model of R&D network formation in Cournot oligopoly, and in Grandjean et al. (2013)'s model of network formation in a Tullock contest.

The structure of stable and efficient networks is in general different. There is a tension between the networks that are formed by the agents and those that would produce the highest sum of payoffs. In a stable network, competitors cooperate with equally connected agents while the sum of payoffs would be higher if the links were more unequally distributed. In Goyal and Joshi (2003)'s model of R&D network formation for example, firms with more partners produce more since they have a smaller marginal cost. The benefit of a new partnership is thus increasing in the degree of a firm since it affects a larger volume of production. Firms in the large clique do not cooperate with those in the small one, and as such do not exploit completely the R&D network benefits, leading to the aforementioned inefficiencies.

Our theoretical predictions mirror findings in the empirical literature on cooperation among rivals. Hochberg et al. (2010) show that networking may create barriers to entry for the supply of venture capital. Regibeau and Rockett (2011) indicate that cross-licensing agreements may warrant antitrust scrutiny. Bekkers et al. (2002) have documented the successful attempt of Motorola in the eighties to create a group of 5 dominant firms in the GSM industry by forming cross-licensing agreements with these firms and refusing agreements with outsiders. Motorola and its competitors have influenced the market structure and ended up dominating the GSM industry. Howard (2009) provides another striking example in the seeds industry, where six of the largest nine firms have closely cooperated through cross-licensing agreements while the three others have formed joint ventures to share research output and expertise. The following picture, taken and adjusted from Howard (2009), summarizes the situation in 2013.

[3]Nested split graphs are networks such that each agent is connected to other agents with less links. The network structures are presented in Figure 2.

[4]Core-periphery networks are networks where some agents (those in the core) are connected to every other agent, while others (those in the periphery) are only connected to agents in the core.

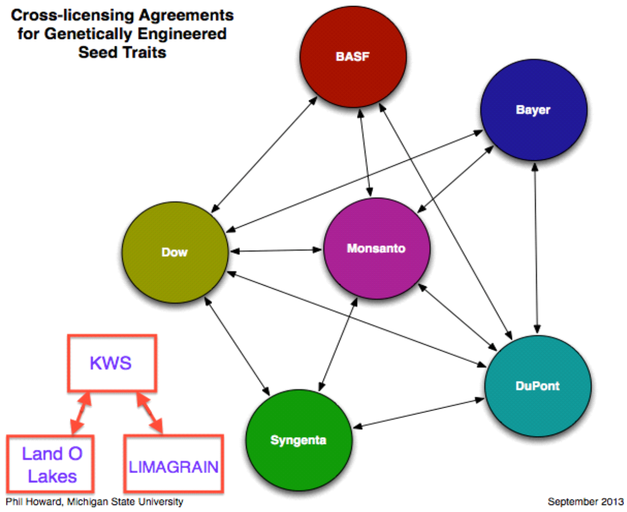

Figure 1: Cross-licensing agreements and joint ventures in the seeds industry

Cooperation among rivals has been studied in a coalition formation setting.[5] Bloch (1995) shows that firms form two asymmetric coalitions in the cost reducing R&D Cournot model, where the largest group contains 3/4 of the firms. Yi (1997) identifies conditions leading to the formation of two asymmetric coalitions in the coalitional unanimity game of Bloch (1996).

Our properties, restricted to networks composed of strongly connected components are stronger than those of Yi (1997). Thus, forward looking agents forming coalitions according to the rules of Bloch (1996)'s coalitional unanimity game would form two coalitions. We find that the farsighted stable set is composed of networks featuring two groups, a strongly connected component among a majority of the agents, and another group of agents that are either strongly connected or not connected. Furthermore, the size of the groups is equivalent in the two approaches. We have identified sufficient conditions for establishing an equivalence between the networks formed by farsighted agents and the coalitions formed among forward looking agents. These conditions are also necessary. By means of examples, we show that the equivalence no longer holds when strong degree monotonicity or minority economies to scale are violated.

In a network formation setting, farsightedness has been shown to lead to an asymmetric partition of the agents in the work of Roketskiy (2012) and of Mauleon et al. (2014). In the model of Roketskiy (2012), the payoff of agents is the sum of two terms. The first is his production, which is increasing in degree, and the second is a bonus shared among the agents having the highest degree. Mauleon et al. (2014) study cost reducing R&D agreements, assuming that R&D externalities perfectly spill over the network so that each member of a component has the same marginal cost, as in a coalition.

Westbrock (2010) study efficient networks by extending the R&D collaboration model of Goyal and Joshi (2003) to a network game of differentiated oligopoly and finds that when

[5] See Bloch (2005) for a survey of this literature.

the participation constraints are not binding, the efficient and profit maximizing networks are interlinked stars.[6] Our focus is on a class of games that includes the model of Goyal and Joshi (2003). Our predictions are narrower than those of Westbrock (2010), and we also analyze the case where participation constraints are binding. König et al. (2012) study R&D collaborations with network dependent indirect spillovers and show that the efficient network structure is a nested split graph. In a standard linear quadratic utility function with local synergies (Ballester et al., 2006), Belhadj et al. (2013) show that an efficient network must be a nested split graph in network games with strategic local complementarity.

The paper is organized as follows. In Section 2 we present our framework and introduce the notation. In Section 3, we provide two motivating examples. In Section 4 and 5, we study respectively pairwise and farsighted stability. Section 6 characterizes the efficient network. Section 7 concludes.

2 Notation and framework

2.1 Networks

Let $N = \{1, 2, ..., n\}$ be a finite set of agents. We write $g_{i,j} = 1$ when a link between i and j exists and $g_{i,j} = 0$ otherwise. A network $g = \{(g_{i,j})_{i,j \in N}\}$ is the list of pairs of individuals who are linked to each other. Let g^N be the collection of all subsets of N with cardinality 2, so g^N is the complete network. The set of all possible networks on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. The network obtained by adding the link ij to an existing network g is denoted $g + ij$ and the network that results from deleting the link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \in N \mid \exists j \text{ such that } ij \in g\}$ be the set of agents who have at least one link in the network g. Let $N_i(g)$ be the set of agents that are linked to $i : N_i(g) = \{j \in N \mid ij \in g\}$. The degree of agent i in a network g is the number of links that involve that agent: $n_i(g) = \#N_i(g)$.[7] A path in a network $g \in \mathbb{G}$ between i and j is a sequence of agents $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j. A component h of a network g is a nonempty subnetwork $h \subseteq g$ satisfying (i) for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j, and (ii) for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. Given a network g, let $K_1(g) = \{i \in N \mid n_i(g) \geq n_j(g) \text{ for all } j \in N\}$ be the set of agents with the highest degree. For all $t \geq 2$, let $K_t(g) = \{i \in N \mid n_i(g) \geq n_j(g) \text{ for all } j \in N \setminus (K_s(g))_{s<t}\}$ be the set of agents with the highest degree among the agents that are not in $K_1(g), ..., K_{t-1}(g)$. We write $X \leftrightarrow_g Y$ if there is at least a link between one agent from the agent set X and one agent from

[6] Interlinked star networks are such that each agent with the maximal number of links is connected to each connected agent.

[7] Throughout the paper we note the cardinality of a set X by the lower case letter x.

the agent set Y in the network g. Similarly, we write $X \top_g Y$ if each agent in X is connected to each agent in Y in g, and $X \perp_g Y$ if there are no links among agents from those sets in g. Let $N^-(g) = \{i \in N(g) \mid n_i(g) \leq n_j(g) \text{ for all } j \in N(g)\}$ be the set of agents in g with the smallest degree among those that are connected, and let $N^0(g)$ be the set of agents that are not connected in g. For $S \subseteq N$, let $g_{-S} = \{jk \in g \mid j \notin S \text{ and } k \notin S\}$ be the set of links among the agents outside S in the network g.

We now define some networks that play an important role in our analysis (see Figure 2). Given a set of agents $S \subsetneq N$, a *dominant group network* network g^S is such that the agents in S are connected to each other in S while the agents in $N \backslash S$ have no links. In a *k-clique network* $g = g^{S_1} \cup ... \cup g^{S_k}$, the agents are partitioned into k groups such that there is a link between every pair of agents in the same group and no link between any two agents in different groups.[8] We write a *2-cliques network* with a clique S involving the majority of the agent by $\widetilde{g}^S = g^S \cup g^{N \backslash S}$. A network g is a *nested split graph* with t classes if $K_s(g) \top_g K_r(g)$ for all $r \leq t-s+1$. The agents in class 1 are connected to every connected agent, while the agents in class t are only connected to the agents in class 1. Similarly, the agents in class 2 are connected to every connected agent other than those in class t while agents in class $t-1$ are only connected to the agents in class 1 and 2, etc. In the nested split graph depicted in Figure 2, there are four classes of agents. A line between two groups indicates that each agent from one group is connected to every agent from the other group. In a *core-periphery network*, each agent in the core is connected to every other agent while agents in the periphery are only connected to agents in the core. Finally, each agent has the same degree in a *regular network*.

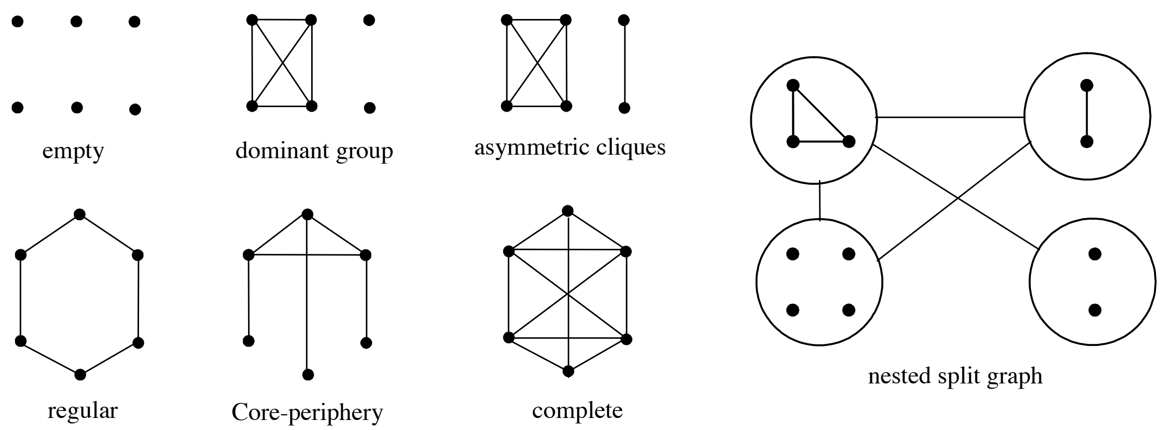

Figure 2: Networks

[8] A clique is a set of agents $S \subseteq N$ such that there is a link between each pair of agents in S. It is maximal if no superset of S is a clique.

2.2 Framework

The games of network formation among rivals $\mathcal{G}$ we consider involves n ex-ante identical agents playing a two-stage game. Agents first form bilateral agreements in the network formation stage. A network g induces a degree distribution of the agents $(n_1(g), n_2(g), ..., n_n(g))$, where the degree of an agent represents its strength, and is the only payoff relevant network statistic. Each agent i then chooses a strategy $\sigma_i \in \sum$ in the competition stage. The strategy set $\sum$ is identical for each agent in every network, it does not depend on the network and it contains the strategy "out". An agent choosing the strategy "out" is guaranteed to get a payoff of 0. We assume that for each network g, there is a unique Nash equilibrium of the game in the second stage $\sigma^*(g) = (\sigma_1^*(g), \sigma_2^*(g), ..., \sigma_n^*(g))$, and we denote by $\pi_i(g)$ the Nash equilibrium payoff of agent i, gross of linking costs. In a network g, the agents playing a strategy other than "out" are the active (or participating) agents $K(g)$. A participating agent gets a positive payoff at the Nash equilibrium since he would otherwise have a profitable deviation, $\pi_i(g) > 0$ iff $i \in K(g)$.

Two properties relate the payoff of an agent to specific network configurations. The first imposes that the stronger an active agent, the higher his payoff, a property called degree monotonicity.

Property 1. Degree monotonicity: $\pi_i(g') > \pi_i(g)$ if $n_i(g') > n_i(g)$, $n_k(g') = n_k(g)$ for all $k \in N \backslash \{i\}$ and $\pi_i(g') > 0$.

The second imposes that the stronger an active agent, the smaller his rivals' payoffs, a property called negative externalities.

Property 2. Negative externalities: $\pi_j(g') < \pi_j(g)$ if $n_i(g') > n_i(g)$ for some $i \neq j$, $n_k(g') = n_k(g)$ for all $k \in N \backslash \{i\}$, $\pi_j(g) > 0$ and $\pi_i(g') > 0$.

These two properties imply that the payoff of agent i is higher than the payoff of agent j in a network g if the degree of i is higher than the degree of j, $n_i(g) > n_j(g) \Longrightarrow \pi_i(g) \geq \pi_j(g)$. Indeed let g' be such that $n_i(g') = n_j(g)$ and $n_k(g') = n_k(g)$ for $k \in N \backslash \{i\}$. Then, we have $\pi_i(g) > \pi_i(g')$ by degree monotonicity, $\pi_i(g') = \pi_j(g')$ by symmetry and uniqueness of the Nash equilibrium, and $\pi_j(g') > \pi_j(g)$ by negative externalities. Degree monotonicity and negative externalities imply that the degree of nonparticipating agents is smaller than the degree of participating agents. We let $K_m(g)$ be the set of weak agents, i.e. the set of participating agents with the smallest degree in the network g, $K_m(g) = \{i \in K(g) \mid n_i(g) \leq n_j(g) \text{ for all } j \in K(g)\}$. The set of nonparticipating agents in g is $E(g) = N \backslash K(g)$ and the set of strong agents, i.e. agents who have strictly more connections than the weak agents is $K^+(g) = K(g) \backslash K_m(g)$.

We assume that each link costs $c \geq 0$ to the two agents involved in the link. The net payoff of agent i is then given by $\Pi_i(g) = \pi_i(g) - cn_i(g)$. We analyze the structure of stable and efficient networks in the class of games $\mathcal{G}$ when some additional properties are satisfied.[9] The first two properties determine the effect of having more collaborations on payoff.

Property 3, *Strong degree monotonicity,* imposes that participating agents who have the same degree find it worthwhile to see their degree increase.

Property 3. Strong degree monotonicity: $\Pi_i(g) < \Pi_i(g')$ for $i \in S \subseteq K_l(g) \subseteq K(g)$ where g' is such that $n_j(g') = n_j(g) + 1$ for all $j \in S$ and $n_j(g') = n_j(g)$ for all $j \in N \setminus S$.

Strong degree monotonicity is stronger than degree monotonicity since it requires that the benefit of increasing one's degree outweighs the additional linking cost and the cost of facing stronger competitors. This property implies that, everything else equal, more cooperation among equals is better for them. Property 4, *Minority economies to scale,* imposes that the payoff of agents in a small clique increases in the size of the clique when they are facing another clique with the majority of agents.

Property 4. Minority economies to scale: $\Pi_i(g^S \cup g^T) < \Pi_i(g^S \cup g^{T \cup \{j\}})$ for $i \in T$ if $T \subseteq K(g^S \cup g^T)$, where $j \notin \{S \cup T\}$, $T \cap S = \emptyset$ and $s \geq n/2$.

Properties 5 and 6 determine the effect of a reallocation of links leading to an increase of the degree of one agent at the expense of a less connected agent. When a network g' can be obtained from a network g by a mean preserving spread of links favouring i at the expense of j, we say that g' is obtained from g by a switch in favor of i relative to j, and write it $g' \in S(g, i, j)$.

Definition 1. A network g' is obtained from g by a **switch** in favor of i relative to j -$g' \in S(g, i, j)$- if $n_i(g') = n_i(g) + 1$, $n_j(g') = n_j(g) - 1$ where $n_i(g) \geq n_j(g)$ while $n_k(g) = n_k(g')$ for all $k \in N \backslash \{i, j\}$.

A switch leads to a new pattern of collaboration where the number of partners of agents are less equally distributed. Property 5, *welfare improving switch,* imposes that the sum of payoffs in a network increases after a switch if the agent whose degree decreases remains active.

Property 5. Welfare improving switch: $\sum_{i \in N} \Pi_i(g') > \sum_{i \in N} \Pi_i(g)$ if $g' \in S(g, i, j)$ and $j \in K(g) \cap K(g')$.

Finally Property 6, *switch externality,* imposes that the payoff of an agent not involved in a switch among strong agents does not decrease.

Property 6. Switch externality: $\pi_l(g') \geq \pi_l(g)$ for $g' \in S(g, i, j)$ if $j \in K^+(g)$ and $l \neq j$.

In the rest of the paper, we show that Properties 1-6 are satisfied in standard models of bilateral cooperation among rivals, and analyze how they shape the farsighted stable set of networks, and the set of efficient networks.

[9] Goyal and Joshi (2006a) and Hellman and Landwher (2014) also propose properties on the payoff function in adjacent networks, and relate these to the structure of the pairwise stable networks.

3 Motivating examples

We show in this section that Properties 1-6 are satisfied in Goyal and Joshi (2003)'s model of bilateral R&D agreements among Cournot competitors and in Grandjean et al. (2013)'s model of cooperation among rivals in a Tullock contest.

3.1 R&D cooperation in the Cournot model

In Goyal and Joshi (2003), n firms first form bilateral R&D agreements to reduce their marginal cost, and then compete à la Cournot. The marginal cost of a firm i depends linearly on its degree $c_i(g) = \lambda - \mu n_i(g)$, where λ is the marginal cost of an isolated firm, and μ measures the effect of links on marginal cost. Firms compete in quantity in a market for homogenous products with the following linear inverse demand function $p = \alpha - \sum\limits_{i \in N} q_i$, where $\alpha > \lambda$ is the size of the market. The Nash equilibrium in the second stage is uniquely given by $q_i(g) = \max\{0, \frac{(\alpha-\gamma_0)+k(g)\gamma n_i(g)-\gamma\sum_{j\in K(g)\setminus\{i\}} n_j(g)}{k(g)+1}\}$, and gross payoffs are $\pi_i(g) = q_i(g)^2$.[10] From the expression characterizing the optimal quantity, one observes that output and thus profit are increasing in own degrees (P1)[11] and decreasing in competitors' degrees (P2). For sufficiently small linking costs, we show that Properties 3-6 are also satisfied.

Lemma 1. *The Cournot model with linear cost reducing R&D $c_i(g) = \lambda - \mu n_i(g)$ and inverse demand function $p = \alpha - \sum\limits_{i \in N} q_i$ satisfies P3-P6 when linking costs are small.*

All proofs are in the appendix. Only own degree and the sum of firms' degrees are payoff relevant in this game. When participation constraints are not binding, the distribution of firms' degrees affects the allocation of production among the competitors but does not affect the total output. After a switch, the output of the firms whose degree remains constant are thus unchanged. It follows that a switch increases the industry profits since the production does not change but the total costs of production have decreased as some units are transfered from the firm whose degree decreases to the firm whose degree increases, and the latter produces the good at smaller marginal costs (P5). When more collaborations are formed, the profit of a firm increases if its degree increases in the same proportion than others.

It follows that strong degree monotonicity and minority economies to scale are satisfied, because the number of news links of competitors is at most $n-1$ in the first case and $2s-1$ in the second, where $s < n/2$ is the size of the minority clique.

3.2 Bilateral agreements in the Tullock contest

Grandjean et al. (2013) consider n agents who participate in a contest to win a prize. Each contestant i chooses the level of effort e_i to make. The marginal cost of effort is unitary $C(e_i) = e_i$.

[10] Goyal and Joshi (2003) assume that firms always produce positive quantities, that is they assume that $(\alpha - \lambda) > (n-1)(n-2)\mu$. Goyal and Moraga (2001), Deroian and Gannon (2008), Westbrock (2010), Mauleon et al. (2014) also analyze this model, and rule out the issue of participation.

[11] We sometimes write Px to refer to Property x.

The prize is allocated to the individuals according to the profile of efforts of all agents according to the contest success function $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$. The valuation of an agent for the good is decomposed into a fixed component v, and a variable component that depends on the degree of the agent : $v_i(g) = v + n_i(g)\beta$. The Nash equilibrium in the second stage is given by $e_i^*(g) = \max\{0, \frac{k(g)-1}{k(g)} h_{k(g)}(g)(1 - \frac{k(g)-1}{v_i(g)} \frac{h_{k(g)}(g)}{k(g)})\}$, where $h_{k(g)}(g) = k(g)/(\sum_{j \in K(g)} 1/v_j(g))$ is the harmonic mean of the largest $k(g)$ valuations.[12] It follows that gross payoffs are given by $v_i(g)(e_i^*(g)/\sum_{j \in N} e_j^*(g))^2$. Effort and thus payoff is increasing in own degrees and decreasing in competitors' degrees, so that this game belongs to $\mathcal{G}$. For sufficiently small linking costs, we show that it also satisfies strong degree monotonicity, minority economies to scale, welfare improving switch and switch externality.

Lemma 2. *The Tullock contest with contest success function $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$, linear cost of effort $C(e_i) = e_i$, and valuation $v_i(g) = v + n_i(g)\beta$ satisfies $(P1) - (P4)$ for small linking costs.*

When participation constraints are not binding, the intuition for this result is as follows. The sum of efforts of competitors is highest the more equally the degree are distributed. The payoff of an agent not involved in a switch then increases after a switch since the same effort leads to higher chances of winning the prize (switch externality). The sum of efforts is lower after a switch, and the expected valuation of the agent getting the prize increases. These two effects leads to higher welfare (welfare improving switch). When equal contestants form new collaborations, their valuations and their chance of getting the prize increases (strong degree monotonicity). When the size of a minority clique increases, each member of the clique gets the prize with higher probability even if the new clique member increases his effort more than the others (minority economies to scale).

4 Pairwise stable networks

Jackson and Wolinsky (1996) have introduced the notion of pairwise stability to characterize the networks immune to a single link addition or deletion. A network is pairwise stable if no agent benefits from severing one of her links and no two agents benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 2. A network g is pairwise stable if
(i) $\Pi_i(g) \geq \Pi_i(g - ij)$ and $\Pi_j(g) \geq \Pi_j(g - ij)$ for each $ij \in g$.
(ii) $\Pi_i(g + ij) > \Pi_i(g)$, then $\Pi_j(g + ij) < \Pi_j(g)$ for each $ij \notin g$.

In a game $\Gamma \in \mathcal{G}$ satisfying strong degree monotonicity, each participating agent with the same degree is connected in a pairwise stable network. Let G^{PS} be the set of pairwise stable

[12] Hillman and Riley (1989) show that the participation of all agents is not guaranteed when the valuations of agents is too asymmetric.

networks. Let $G^* = \{g \in \mathbb{G} \,|ij \in g$ if $i, j \in K_t(g) \subseteq K(g)$, and $ij \notin g$ if $\{i, j\} \nsubseteq K(g)\}$ be the set of networks such that each participating agent with the same degree is connected, and where nonparticipating agents are not connected. A pairwise stable network belongs to this set.

Proposition 1. *Let $\Gamma \in \mathcal{G}$. We have $G^{PS} \subseteq G^*$ if A1 holds.*

A pairwise stable network is composed of cliques of agents having the same degree. Agents in different cliques do not have the same degree. Also, each agent in a clique has the same number of links towards agents in other cliques. The complete network is always pairwise stable. If an agent deviates from the complete network by cutting a link, he either reaches a network g' where he is not participating or where he is participating but not connected to some agents with the same degree. In both cases he is better off by maintaining his links. Dominant group networks fall into this class provided isolated agents do not participate. A dominant group network is pairwise stable if in addition two isolated agents do not participate by forming a link. Network formation can thus endogenously create a barrier to entry for ex ante symmetric agents. Dominant group networks are the only pairwise stable network in the linear Cournot or Tullock model since any two participating agents are better off by forming a link in these games, not only those with the same degree. A network composed of completely connected components of different sizes is also in G^*. Such a network is pairwise stable if agents in different components are not better off by forming a link.

5 von Neumann-Morgenstern farsighted stability

In this section, we analyze the formation of networks among rivals when agents are farsighted, i.e. they anticipate how other agents would react to their choice of partners. We use the notion of indirect dominance of Harsanyi (1974) to account for farsighted behavior. A network g *indirectly dominates* a network g' if there exists a sequence of networks that implements g over g' such that in every network in the sequence g_k, all deviating agents have a higher payoff in the end network g than in the current network g_k. In a network g_k in the sequence from g' to g, any group of agents $S \subseteq N$ may *enforce* the network g_{k+1} over g_k if the links that are created involve two agents from S while those that are deleted involve at least an agent from S.

Formally, enforceability and indirect dominance are defined as follows.

Definition 3. Given a network g, a coalition $S \subseteq N$ is said to be able to **enforce** a network g' if

(i) $ij \in g$ but $ij \notin g' \Longrightarrow \{i, j\} \cap S \neq \emptyset$
(ii) $ij \notin g$ but $ij \in g' \Longrightarrow \{i, j\} \subseteq S$

Definition 4. A network g is **indirectly dominated** by a network g', or $g \ll g'$, if there exists a sequence of networks $g_0, g_1, ..., g_T$ (where $g_0 = g$ and $g_T = g'$) and a sequence of coalitions $S_0, S_1, ..., S_{T-1}$ such that for any $t \in \{1, 2, ..., T\}$,

(i) $\pi_i(g_T) > \pi_i(g_{t-1})$ for all $i \in S_{t-1}$, and

(ii) coalition S_{t-1} can enforce the network g_t over g_{t-1}.

We use the notion of indirect dominance in the stable set of von Neumann and Morgenstern (1944). A farsighted stable set of networks is such that no network in the set indirectly dominates another network in the set (internal stability) and each network not in the set is indirectly dominated by a network in the set (external stability). In this sense, a deviation from a stable network leading to a network outside the set is not accounted for since the network reached is itself unstable.

Definition 5. A set of networks $G \subseteq \mathbb{G}$ is a **von Neumann-Morgenstern farsighted stable set** of a game $\Gamma \in \mathcal{G}$ if
(i) for all $g \in G$, there does not exist $g' \in G$ such that $g \ll g'$, and
(ii) for all $g' \notin G$, there exists $g \in G$ such that $g' \ll g$.

We show that a von Neumann-Morgenstern farsighted stable set always exists in a network formation game among rivals satisfying strong degree monotonicity and minority economies to scale. Let $\widetilde{s} \in \{1, ..., n\}$ be the minimal size of a large clique S such that the remaining agents do not participate in the clique network $\widetilde{g}^S$: $K(\widetilde{g}^S) = S$ and $K(\widetilde{g}^{S \setminus \{k\}}) = N$ for $k \in S$. Farsighted agents either form one large clique to drive the remaining agents out of the market, or they form a smaller one to reduce the number of strong competitors and accommodate full participation. By excluding a member from the large clique, each strong competitor becomes weaker but is facing fewer strong competitors. When the size of the large clique is smaller than $\widetilde{s}$, the excluded member joins the smaller clique so that each weak competitor becomes stronger. We denote by $\widehat{s} \in \{\widetilde{s}, ..., n\}$ the size of the large clique that maximizes the per capita value $\widehat{\pi}$ of its members when the remaining agents are excluded, and by $\overline{s} \in \{1, ..., \widetilde{s} - 1\}$ the size of the large clique that maximizes the per capita value of its members $\overline{\pi}_1$ when the remaining are accommodated and form a smaller clique. For $i \in S$, we have $\widehat{s} \in \arg\max_{s \in \{\widetilde{s}, ..., n\}} \Pi_i(g^S)$, $\widehat{\pi} = \max_{s \in \{\widetilde{s}, ..., n\}} \Pi_i(g^S)$, $\overline{s} \in \arg\max_{s \in \{1, ..., \widetilde{s}-1\}} \Pi_i(\widetilde{g}^S)$ and $\overline{\pi}_1 = \max_{s \in \{1, ..., \widetilde{s}-1\}} \Pi_i(\widetilde{g}^S)$.[13]

In Proposition 2, we show that the set of dominant group networks of size $\widehat{s}$ is a von Neumann-Morgenstern farsighted stable set if and only if $\widehat{\pi} > \overline{\pi}_1$ when the game satisfies strong degree monotonicity.

Proposition 2. *Let* $\Gamma \in \mathcal{G}$ *satisfy Property* 1*. Then,* $G^{FS1} = \{g^S \mid s = \widehat{s}\}$ *is a vNM farsighted stable set iff* $\widehat{\pi} > \overline{\pi}_1$.

The intuition for Proposition 2 is as follows. No network in the set g^T is indirectly dominated by another $g^{T'}$ since in every path from g^T to $g^{T'}$, each member of T has a payoff of $\widehat{\pi}$ in the first network g' where some members of T modify the network, and thus does not benefit from the deviation. Thus internal stability is satisfied. Also, each network g' outside the set is indirectly dominated by a network g in the set. To see this, let us construct a path from g' to g where at each step in the path, agents with a payoff smaller than $\widehat{\pi}$ delete their links until a network g''

[13] For simplicity, we assume that $\arg\max_{s \geq \widetilde{s}} \Pi_i(g^S)$ and $\arg\max_{s \leq \widetilde{s}-1} \Pi_i(\widetilde{g}^S)$ are singleton sets.

is reached where either $\widehat{s}$ agents are isolated, or each connected agent has a payoff greater than $\widehat{\pi}$. In the first case, unconnected agents form a complete component, leading to the exclusion of the remaining agents who then delete their links to save linking costs. In the second case, there are less than $\widetilde{s}$ connected agents since the payoff of a connected agent with the smallest degree in a network with $s \geq \widetilde{s}$ connected agents is smaller than in a clique among s agents -and is thus smaller than $\widehat{\pi}$- by negative externalities and strong degree monotonicity. In g'', unconnected agents create links until they all have the same degree as an agent i with the lowest degree in g'', or until they are completely connected. In the first case, agent i has the lowest degree at the current network. He would be better off in the complete network - and thus in the end network- by negative externalities and strong degree monotonicity. In the other case, each agent participates since agents with lowest degree have more than $n - \widetilde{s}$ links while the agents with the highest degree have less than $\widetilde{s}$ links. The payoff of agent i is then smaller than $\overline{\pi}_1$ -and thus than $\widehat{\pi}$ by assumption- by negative externalities and strong link monotonicity. At the current network, the agents with no links in g'' and agent i delete their links. The path proposed is such that at each step, at least one additional agent cuts his links. After a finite number of iterations, $\widehat{s}$ agents are isolated and form a complete component. The remaining agents do not participate and delete their useless links. By construction, the path does not reach a dominant group network of size $\widehat{s}$ in an intermediate step, as we would then have agents deleting their links at a network where their payoff is positive and ending up not participating in the final network. It follows that the condition $\widehat{\pi} > \overline{\pi}_1$ is sufficient for the set G^{FS1} to be a vNM farsighted stable set. It is also necessary since a network composed of two cliques of size $\overline{s}$ and $n - \overline{s}$ is not indirectly dominated by a network in the set when $\widehat{\pi} \leq \overline{\pi}_1$.

In Proposition 3, we show that the set of networks composed of two cliques of size $\overline{s}$ and $n - \overline{s}$ is a von Neumann-Morgenstern farsighted stable set if and only if $\widehat{\pi} > \overline{\pi}_1$ when the game satisfies strong link monotonicity and minority economies to scale.

Proposition 3. *Let $\Gamma \in \mathcal{G}$ satisfy Properties 1 and 2. The set $G^{FS2} = \{g \subseteq g^N \mid g = \widetilde{g}^S$ such that $\#S = \overline{s}\}$ is a vNM farsighted stable set if and only if $\overline{\pi}_1 > \widehat{\pi}$,.*

The intuition for Proposition 3 is as follows. No network in the set $\widetilde{g}^T$ is indirectly dominated by another $\widetilde{g}^{T'}$ since in every path from $\widetilde{g}^T$ to $\widetilde{g}^{T'}$, each member of T has a payoff greater than $\overline{\pi}_1$ in the first network g' where some members of T modify the network by negative externalities. Thus internal stability is satisfied. Also, each network g' outside the set is indirectly dominated by a network g in the set. To show this, we construct a path from g' to g where at each step in the path, agents with a payoff smaller than $\overline{\pi}_1$ delete their links until a network g'' is reached where either $\overline{s}$ agents are isolated, or each connected agent has a payoff greater than $\overline{\pi}_1$. In the first case, unconnected agents form a complete component, while the agents with the lowest degree among the remaining agents successively delete their links and then form the second clique. The payoff of an agent who is in the small clique in the end network is smaller than $\overline{\pi}_2$ at a network where he deviates, either by minority economies to scale if he deletes links or by strong degree monotonicity when the second clique is formed. In the second case, unconnected agents create links until they all have the same degree as an agent i with the lowest degree in g'', or until they are completely connected. The payoff of the agents with less links than agent i is smaller than $\overline{\pi}_1$ at the current network by negative externalities and strong degree monotonicity, and delete their links looking forward to the end network where they are in the large clique. It follows that the condition $\overline{\pi}_1 > \widehat{\pi}$ is sufficient for the set G^{FS2} to be a vNM farsighted stable set. It is also necessary since a network composed of one clique of size $\widehat{s}$ is not indirectly dominated by a network in the set when $\widehat{\pi} \geq \overline{\pi}_1$.

Propositions 2 and 3 thus establish existence of a von Neumann-Morgenstern farsighted stable set of networks G^{FS}, where $G^{FS} = G^{FS1}$ if $\widehat{\pi} > \overline{\pi}_1$ while $G^{FS} = G^{FS2}$ if $\widehat{\pi} < \overline{\pi}_1$. The agents are partitioned into two groups $\{S^*, N \setminus S^*\}$ where the size s^* of the large group S^* is $\widehat{s}$ if $\widehat{\pi} > \overline{\pi}_1$ or $\overline{s}$ if $\widehat{\pi} < \overline{\pi}_1$. The partition of the agents in a farsighted stable network is then equivalent to the subgame perfect equilibrium of Bloch's (1996) coalition unanimity game. One could interpret a coalition in the coalition formation approach to be a complete component in a network formation setting. This interpretation is in line with the applications discussed in both literatures, as cooperation matters only through the number of partners of agents. For instance Bloch's (1995) model of group formation and Goyal and Joshi's (2003) model of network formation among agents competing in Cournot both lead to the same profile of marginal costs, and thus to the same equilibrium payoff, when a clique structure in Goyal and Joshi mirrors a group structure in Bloch. The rules of the coalition unanimity game are as follows. Agents are ranked according to an exogenous rule of order. The first agent proposes the formation of a coalition. If all members of this proposed coalition agree, then the coalition is formed and can no longer be dissolved and the game continues in which the first agent in the updated ranking after removing the first coalition, makes the next proposal. If one agent rejects the proposal, he becomes the initiator in the next round. The proposer of a coalition and its potential members must thus foresee the coalition structure that will eventually prevail in order to decide on the current coalitional proposal. Yi (1997) shows that ex ante symmetric agents form the partition $\{S^*, N \setminus S^*\}$ in this game provided (i) when 2 coalitions merge, the remaining agents are worse off, (ii) a member of a coalition is better off if his coalition merges with a larger coalition, (iii) a member of a coalition is better off if he leaves a coalition to join another of larger size, and (iv) members of any coalition of size $s \leq n/2$ do not want to exclude a member. negative externalities and strong degree monotonicity imply that conditions (i)-(iii) are satisfied. Whether negative externalities, strong degree monotonicity and minority economies to scale imply (iv) remains an open question.[14]

We now show that minority economies to scale is necessary for our results. Consider $N = \{1, 2, ..., 6\}$ firms competing in quantities. The marginal cost of an agent i depends on his degree in the network g in the following way: $c_i(g) = c(n_i(g))$ where $c(0) = \frac{1}{2}, c(1) = \frac{44}{100}, c(2) = \frac{43}{100}, c(3) = \frac{42}{100}, c(4) = \frac{41}{100}$ and $c(5) = \frac{81}{200}$. The linear inverse demand curve is $p = 1 - \sum_{i \in N} q_i$.

[14]If not, the partition $\{S^*, N \backslash S^*\}$ could be the outcome of a game of network formation among farsighted agents but not a subgame perfect equilibrium of the coalition unamity game.

The participation constraints never bind and hence:

$$q_i = \frac{1}{n+1}\left[1 - nc(n_i(g)) + \sum_{j\in N\backslash\{i\}} c(n_i(g))\right]$$

The cost of forming a link is equal to $\varepsilon > 0$, where ε is arbitrarily small. This game satisfies negative externalities and strong degree monotonicity. Minority economies to scale is violated since agents in the group of two firms prefer to keep the isolated agent without connections rather than forming a clique with him. Assuming that the marginal cost of a firm in a coalition of size s is given by $c(s-1)$, the equilibrium coalition structure of the coalition unanimity game is a partition of the six agents into a group of 3 agents, another of 2 agents and a singleton. Let $g_1 = g^{S_1} \cup g^{S_2}$ be a network composed of two cliques $S_1 = \{1,2,3\}, S_2 = \{4,5\}$. Abusing notation we write $g_1 = \{123, 45, 6\}$. The set of permutations of g_1 which do not mutually indirectly dominate each other is given by $G_1 = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ where:

$$\begin{aligned}
g_1 &= \{123, 45, 6\} \\
g_2 &= \{123, 46, 5\} \\
g_3 &= \{123, 56, 4\} \\
g_4 &= \{126, 45, 3\} \\
g_5 &= \{136, 45, 2\} \\
g_6 &= \{236, 45, 1\}
\end{aligned}$$

The set G_1 is internally stable. One can see that every other permutation of g_1 is indirectly dominated by a network in G_1. It follows that every von Neumann Morgenstern farsighted stable set G containing solely g_1 and permutations of it is a subset of G_1. But then, G does not satisfy external stability as no network in the set indirectly dominates $g_7 = \{16, 23, 24, 35\}$ for instance. Existence of a vNM farsighted stable set when minority economies to scale is violated remains an open question.

When dominant group networks are farsighted stable $G^{FS} = G^{FS1}$, a farsighted stable network g is also pairwise stable. Indeed, an agent does not find it profitable to delete a link from g by strong degree monotonicity, while an isolated agent does not gain by adding a link since he would remain inactive. When asymmetric cliques are farsighted stable $G^{FS} = G^{FS2}$, a farsighted stable network is pairwise stable if two agents from different cliques do not find it profitable to add a link as connected agents do not want to cut a link by strong degree monotonicity.

6 Efficiency

In this section, we discuss the relationship between the network architecture and welfare for a game $\Gamma \in \mathcal{G}$ satisfying Properties 3 and 4. A network is efficient if it maximizes the sum of payoffs of the agents $W(g) = \sum_{i \in N} \Pi_i(g)$.

Definition 6. A network $g \in \mathbb{G}$ is efficient if $W(g) \geq W(g')$ for all $g' \in \mathbb{G}$.

Switch externalities implies that the set of active agents does not shrink after a switch among strong agents. A switch among strong agents then improves welfare by welfare improving switch. Welfare improving switch is silent on the effect of a switch leading to the exclusion of the agent whose degree decreases. A switch would in that case weaken an almost inactive agent, and would then be almost equivalent as reinforcing a single agent. This could harm welfare in some applications, as is illustrated in the following example.

Example 1. Take the Tullock contest model among $n = 11$ agents. Suppose the common valuation for the good is $v = 0$ and the effect of cooperation on valuation is $\beta = 1$. Let the network g be such that agent 1 is connected to all agents, agent 2 is connected to agents 3 and 4 and agent 3 is connected to agent 5, that is $g = \{i_1i_2, i_1i_3, ..., i_1i_{11}, i_2i_3, i_2i_4, i_3i_5\}$. The valuation of contestants for the prize is given by $v_1 = 10, v_2 = 4, v_3 = 3$ and $v_k \leq 2$ for $k \geq 4$. Then, agents $1, 2$ and 3 participate and get respectively a payoff of $\pi_1(g) = 5,003$, $\pi_2(g) = 0,288$, and $\pi_3(g) = 0,002$. Let the network g' be obtained from g by replacing the link i_3i_5 by the link i_2i_5. Agent 3 does not participate in g' and payoffs are $\pi_1(g') = 4,444$ and $\pi_2(g') = 0,555$ so that the welfare decreases. The allocation of the prize is less efficient in g'. The probability agent 1 gets the prize falls from 0,71 in g to 0,66 in g'.

Nested split graphs have been introduced in applied mathematics by Cvetkovic and Rowlinson (1990) and Mahadev and Peled (1995). Agents in a nested split graph can be decomposed into t classes such that an agent in class s is connected to each agent in class 1 to $t - s + 1$. If the network g is a nested split graph with t classes, then $K_s(g) \top_g K_r(g)$ for all $r \leq t - s + 1$. The agents in class 1 are connected to every connected agent, so that there is at most one component in the network. One easily sees that a network is immune to switches if and only if it is a nested split graph. It follows that the efficient network is a nested split graph if agents are active in every network configuration.

Proposition 4. *The efficient network is a nested split graph if $K(g) = N$ for all $g \in \mathbb{G}$.*

Intuitively, if it was not the case, it would be possible to improve welfare by letting an agent replace a link by another so that his new partner has more connections than the original one.

For a given network g, let $C^-(g)$ be the set of networks g' where the degree distribution of the participating agents does not change while the sum of degrees of nonparticipating agents decreases. Formally, $C^-(g) = \{g' \in \mathbb{G} \mid n_l(g) = n_l(g')$ for all $l \in K(g)$, $\sum_{k \in E(g)} n_k(g) > \sum_{k \in E(g)} n_k(g')$ and $\max_{l \in E(g)} n_l(g) \geq \max_{l \in E(g)} n_l(g')\}$. When agents do not participate in some network configuration, an efficient network is immune to switches among strong agents, and minimizes the linking costs given the degree distribution of the participating agents.

Let $\overline{G}$ be the set of networks satisfying these constraints. Formally,

$$\overline{G} = \{g \in \mathbb{G} \mid \text{(i) } C^-(g) = \{\emptyset\} \text{ and (ii) } \nexists g' \in \mathbb{G} \text{ such that } g' \in S^*(g, i, j) \text{ for } i, j \in K^+(g)\}$$

In a network $g \in \overline{G}$, nonparticipating agents are not connected among each other and the connections among highly connected agents form a nested split graph. Let us decompose the set $\overline{G}$ into three sets $\overline{G} = \overline{G}_1 \cup \overline{G}_2 \cup \overline{G}_3$. Participating agents have the same degree in a network in $\overline{G}_1$. Strong agents form a clique in a network in $\overline{G}_2$. Finally, the nested split graph among strong agents involve more than one class of agents in a network in $\overline{G}_3$. Formally,

$$\begin{aligned}\overline{G}_1 &= \{g \in \overline{G} \mid K_1(g) = K(g)\}\\ \overline{G}_2 &= \{g \in \overline{G} \mid K_1(g) \subsetneq K(g) \text{ and } g_{-N\setminus K^+(g)} = g^{K^+(g)}\}\\ \overline{G}_3 &= \{g \in \overline{G} \mid K_1(g) \subsetneq K(g) \text{ and } g_{-N\setminus K^+(g)} \neq g^{K^+(g)}\}\end{aligned}$$

We now show that networks in $\overline{G}$ are either core-periphery or quasi nested split graphs. Proposition 5 shows that if each participating agent has the same degree in a network $g \in \overline{G}$, then they should all have the same number of links to other participating agents. In addition, they should primarily be connected among each other so as to minimize the linking costs. Networks in $\overline{G}_1$ are regular networks on a set of agents, ranging from the empty to the dominant group network, and core-periphery networks where agents in the core form a clique while agents in the periphery are only connected to those in the core and are not active. Core-periphery networks range from the dominant group network to the nested split graph with two groups.

Proposition 5. *Let $g \in \overline{G}_1$. Let $K(g) = \{i_1, i_2, ..., i_{k(g)}\}$ be such that $n_{i_1}(g_{-E(g)}) \geq n_{i_2}(g_{-E(g)}) \geq ... \geq n_{i_{k(g)}}(g_{-E(g)})$. If $n_{i_1}(g_{-E(g)}) = k(g) - 1$, then $K_1(g) \top K_1(g)$. Otherwise, if $n_{i_1}(g_{-E(g)}) < k(g) - 1$, then $n_{i_1}(g_{-E(g)}) = n_{i_2}(g_{-E(g)}) = ... = n_{i_{k-1}}(g_{-E(g)})$, and $n_{i_k}(g_{-E(g)}) \in \{n_{i_1}(g_{-E(g)}) - 1, n_{i_1}(g_{-E(g)})\}$.*

The intuition for Proposition 5 is as follows. Let a network g in $\overline{G}_1$ be such that the participating agents are not entirely connected among each other. There is then at most one link from a participating agent to a nonparticipating agent as it would otherwise be possible to replace two links by one and keep the degree of participating agents unaffected.[15] If strong degree monotonicity holds, $\overline{G}_1$ is only composed of core-periphery networks. Dominant group and core-periphery networks where the periphery is completely connected to the core are the only nested-split graphs belonging to $\overline{G}_1$. Every other network in $\overline{G}_1$ is not efficient if the set of participating agents is unchanged after a switch in favor of a participating agent at the expense of another.

In a network g in $\overline{G}_2$, the strong agents (those in $K^+(g)$) are completely connected among each other. We show in Proposition 6 that the network g is a nested split graph if weak agents (those in $K_m(g)$) are only connected to strong agents. Otherwise, almost all strong agents are connected to each participating agent, and to nonparticipating agents if there is at

[15] A link between an agent in the core and a peripheral agent may be needed since participating agents each have the same degree. Let $\overline{g} \in \overline{G}_1$ be such that each of 5 participating agents have 3 links, then one agent in the core is connected to a peripheral agent. If $N = \{1, 2, 3, 4, 5, 6\}$, an example of such a network is $\overline{g} = \{13, 14, 15, 23, 24, 25, 34, 56\}$.

least one link between a nonparticipating agent and a weak agent. Some strong agents are connected to all connected agents, while others may only be connected to other strong agents in a nested split graph. There are then at most 7 classes of agents $K_1(g) = T_1\backslash\{i\}$, $K_2(g) = T_1$, $K_3(g) = S_1$, $K_4(g) = K^+(g)\backslash S_1$, $K_5(g) = K_m(g)$, $K_6(g) = E_1(g)$, $K_7(g) = E_2(g)$, where $i \in T_1 \subsetneq S_1 \subseteq K^+(g)$, and $E_1(g) \cup E_2(g) = E(g)$. If the network $g \in \overline{G}_2$ is not a nested split graph, almost all strong agents are at least connected to all participating agent.

Proposition 6. *Let $g \in \overline{G}_2$, then*

(i) If $X \leftrightarrow_g K_m(g)$, where $X \in \{K_m(g), E(g)\}$, then $(K_m(g) \cup X)\top_g K^+(g)\backslash\{i_1\}$

(ii) If $(E(g) \cup K_m(g))\bot_g K_m(g)$, then

(ii.a) $K_m(g)\top_g S_1$ and $K_m(g)\bot_g S_1$ for some $S_1 \subseteq K^+(g)$

(ii.b) $E(g)\bot_g N\backslash T_1$ for some $T_1 \subsetneq S_1$, $E(g)\top_g T_1\backslash\{i\}$ for some $i \in T_1$

The intuition for Proposition 6 is as follows. If weak agents are only connected to strong agents, they should be connected to the same set of agents S_1 as switches would otherwise be possible by changing the partner of a weak agent. For the same reason, nonparticipating agents are connected only to agents that are connected to all the others. A network g in $\overline{G}_2$ may involve connections among the weak agents. In that case almost all weak agents are connected to almost all strong agents to avoid a switch. Indeed, if two weak agents are not connected to two strong agents, a switch could be obtained replacing a link among two weak agent and another link among two strong agents by a link between the two weak agents and the same strong agent. It follows that the strong agent with the lowest degree -i_1- should have some connections towards other agents since he has more links than weak agents. Thus, all weak agents should be connected to almost all strong agents to avoid switches. Indeed, it would otherwise be possible to replace a link involving i_1 by another involving another strong agent. There may also be connections between weak agents and nonparticipating agents in the network g. Nonparticipating agents should then be connected to almost all strong agents. For example, if the participating agents form a clique, then the strong agent with lowest degree i_1 should be connected to nonparticipating agents since he has more links than weak agents. Each nonparticipating agent connected to i_1 should be connected to all strong agent as a switch would otherwise be possible. Thus, if a nonparticipating agent is not connected to a strong agents other than i_1, it would be possible to replace a link involving i_1 by this missing link.

The nested split graph g^* that connects the strong agents in a network $g \in \overline{G}_3$ contains more than one class of agents. Proposition 7 shows that networks in $\overline{G}_3$ are nested split graphs. Weak agents are only connected to strong agents, while nonparticipating agents are only connected to agents that are connected to all participating agents.

Proposition 7. *Let $g \in \overline{G}_3$. Then,*

(i) $K_m(g)\top_g S_1$ and $K_m(g)\bot_g N\backslash S_1$ for some $S_1 \subsetneq K_1(g_{-N\backslash K^+(g)})$

(ii) $E(g)\perp_g N\backslash T_1$ *for some* $T_1 \subsetneq S_1$, $E(g)\top_g T_1\backslash\{i\}$ *for some* $i \in T_1$, *and if* $n_{i_E}(g) > n_{j_E}(g)$ *for* $i_E, j_E \in E(g)$, *then* $E(g)\backslash\{j_E\}\top_g\{i\}$.

The intuition for Proposition 7 is as follows. Let g be a network in $\overline{G}_3$, and suppose g^* is the set of links connecting agents in $K^+(g)$ under the network g. The network g^* is a nested split graph with more than one class of agents. Let $K_{m'}(g^*)$ be the set of agents from $K^+(g)$ with the lowest degree in g^*. Agents from this class are connected to those in $K_1(g^*)$, and not to those in $K_2(g^*), ..., K_{m'}(g^*)$ since g^* is a nested split graph. Since they are not connected among each other, they cannot be connected to nonparticipating agents. They are not connected either to the weak agents as those should then be connected to all agents in $K^+(g)$, and a switch would then be possible.[16] The weak agents are not connected to all the agents in $K_1(g^*)$ as they would then have a higher degree than agents in $K_{m'}(g^*)$. In turn, this implies that they are connected to a subset S_1 of agents in $K_1(g^*)$, and not to others. If nonparticipating agents have some connections, they are completely connected to some agents T_1 who are connected to all the other agents.

The efficient networks of a game in $\Gamma \in \mathcal{G}$ satisfying Properties 3 and 4 belong to the set of network $\overline{G}$. Networks in $\overline{G}$ other than nested split graphs are not immune to switches at the expense of a weak agent. They are thus not efficient if the agent whose degree is reduced by a switch remains active. Otherwise a network in $\overline{G}$ may not be efficient because another network in $\overline{G}$ generates a higher welfare.

When agents are active in every network configuration, the complete network and the dominant group network of size $n-1$ are the only networks that may be both stable and efficient. Efficient networks are nested split graphs. If there is more than one class of agents in an efficient network, the agents in the lower classes are not connected but they have the same degree and thus would be better off by forming a link by strong degree monotonicity. In a stable network with more than one clique, welfare could be increased by changing the partner of an agent in a small clique.

If agents may prefer not to participate, inactive agents are not connected in a stable network, while they could be connected to the most connected agents in an efficient network. Stable asymmetric cliques are inefficient. Indeed, welfare could be improved by cutting a link in the two cliques, and adding two links between the agents in the small clique that are not connected and an agent in the large clique.

7 Conclusion

In this paper, we have studied the formation of bilateral agreements when cooperation between pairs of agents creates negative externalities on the remaining ones. This occurs for example

[16] To see this, notice that if weak agents are connected to agents in $K_1(g^*)$, they cannot be completely connected in g as they would otherwise have more links than agents in $K_{m'}(g^*)$. It is then possible to form a link among weak agents, cut the links between those agents and agent i in $K_{m'}(g^*)$, and add a link between i and some other agent j in $K^+(g)$ to whom he is not connected, leading to a network $g' \in S(g, j, i)$.

when firms share patents through cross licencing agreements or share the cost of joint R&D projects, when countries sign bilateral trade agreements, etc. In these applications, the number of competitors is usually rather small and the stakes at hand are important. This motivates us to depart from the standard stability notions in network formation that assume that agents are myopic. Rather, we analyze the networks formed by farsighted agents, that is by agents who forecast how other agents would react to their choice of partners, and take a decision by comparing the current network to the end network that forms when other agents have further deviated. We use the notion of von Neumann Morgenstern farsighted stable set, which can be interpreted as a standard of behavior when agents are farsighted.

We show that there always exists a farsighted stable set in a game of network formation among rivals satisfying *strong degree monotonicity* and *minority economies to scale*. It is either composed of dominant group networks, where isolated agents are excluded from the market, or of networks composed of two asymmetric cliques. Our results thus support two empirically relevant properties of observed R&D and cross-licensing networks: barriers to entry and clustering.

We then show that the efficient network is a nested split graph when the game satisfies *welfare improving switch* if agents are active in every network. Otherwise, if agents may prefer to leave the market in some network configurations, the efficient networks are either (quasi-)nested split graphs or core-periphery networks when *welfare improving switch and switch externality* are satisfied. As a result, the structure of stable and efficient networks is in general different, resulting in a tension between the networks that are formed by the agents and those that would produce the highest sum of payoffs.

The four properties we impose are satisfied in many models of network formation among rivals. We show it is the case in Goyal and Joshi (2003)'s model of R&D network formation in Cournot oligopoly, and in Grandjean et al. (2013)'s model of network formation in a Tullock contest.

We conclude this paper with some directions for future research. First, we have identified one farsighted stable set out of possibly many. We do not know at this stage whether other candidates exist, and if some exist, identifying all the candidates is probably not a realistic task. One could restrict the candidates to be considered, for example by only considering the sets composed of one network and its permutations. One could also analyze whether our properties could be strenghtened to guarantee that our candidate is unique.

Second, one could go in the other direction and study which networks would form if our properties were weakened. In particular, one could ask whether a set composed of a k-cliques network and its permutations could be farsightedly stable if minority economies to scale were not satisfied.

Third, it would be interesting to analyze the case of positive externalities, where the formation of an agreement between two agents benefit the other agents. This occurs for instance in Belleflamme and Bloch (2004)'s model of market sharing agreements, where firms may commit not to compete in each other's market, thereby reducing competition in these markets and increasing the profit of outsiders.

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8 Appendix

Proof of Lemma 1.

(P1) Let $g \in \mathbb{G}$. Let $S \subseteq K_l(g) \subseteq K(g)$. For all $i \in S$, let $n_i(g') = n_i(g) + 1$ and for all $i \notin S$, let $n_i(g') = n_i(g)$. We show that $\pi_i(g') - \pi_i(g) = q_i(g')^2 - q_i(g)^2 > 0$ for $i \in S$ if $i \in K(g')$.

Let us write $q_i(g')^2 - q_i(g)^2 = (q_i(g') - q_i(g))(q_i(g') + q_i(g))$. We show that $q_i(g') - q_i(g) > 0$. Notice that if $S \subseteq K(g)$, then $K(g') \subseteq K(g)$.

By negative externalities, we have $K(g') \subseteq K(g)$.

(i) If $K(g) = K(g')$, then $q_i(g') - q_i(g) = \gamma(k(g) - (s-1))/(k(g) + 1) > 0$

(ii) If $K(g') \subsetneq K(g)$, then let Π_i be the unique Nash equilibrium payoff of agent i in a game among the agents in $K(g')$ who have the same marginal cost as in the network g. By negative externalities, $\Pi_i > \Pi_i(g)$ for all $i \in K(g')$. From step 1, we know that $\Pi_i(g') > \Pi_i$.

(P2) Let $g = g^T \cup g^S$, where $T \cap S = \emptyset$, $T \cup S \subsetneq N$, and $s \leq (n-1)/2 < t$. We show that if $S \subseteq K(g)$, then $\Pi_i(g') > \Pi_i(g)$ where $g' = g^T \cup g^{S\cup\{j\}}$ and $g = g^T \cup g^{S\cup\{j\}}$ for $i \in S$, $j \in N\backslash\{S \cup T\}$.

We have $\pi_i(g') > \pi_i(g)$ if

$$\frac{(\alpha - \gamma_0) + k(g')\gamma s - t(t-1)\gamma - s^2\gamma}{k(g') + 1} > \frac{(\alpha - \gamma_0) + k(g)\gamma(s-1) - t(t-1)\gamma - (s-1)^2\gamma}{k(g) + 1}$$

(i) Suppose that $K(g) = K(g')$. Then, $\pi_i(g') > \pi_i(g)$ if

$$
\begin{aligned}
k(g)s - t(t-1) - s^2 &> k(g)(s-1) - t(t-1) - (s-1)^2 \\
s &< t+1
\end{aligned}
$$

(ii) Suppose that $K(g) = N$ and $K(g') = T \cup S \cup \{j\}$. Then, $\pi_i(g') > \pi_i(g)$ if

$$
\begin{aligned}
& \frac{1}{t+s+2}\left[(\alpha - \gamma_0) + (t+s+1)\gamma s - t(t-1)\gamma - s^2\gamma\right] \\
> \; & \frac{1}{n+1}\left((\alpha - \gamma_0) + n\gamma(s-1) - t(t-1)\gamma - (s-1)^2\gamma\right)
\end{aligned}
$$

which can be rewritten as

$$
\frac{1}{t+s+2}\left[\left(\frac{n-t-s-1}{n}\right)\left[(\alpha-\gamma_0) - t(t-1)\gamma + \gamma s - s^2\gamma\right] + \left(\frac{n-t-s-1}{n}\right)\left[n\gamma - (2s-1)\gamma\right]\right] > 0
$$

Since $(\alpha - \gamma_0) - t(t-1)\gamma + \gamma s - s^2\gamma > 0$, $\pi_i(g') > \pi_i(g)$ when $n\gamma - (2s-1)\gamma > 0$, which is satisfied when $2s < n+1$.

(iii) Suppose that $K(g) = S \cup T$ and $K(g') = T \cup S \cup \{j\}$. Then, $\pi_i(g') > \pi_i(g)$ if

$$
\begin{aligned}
& \frac{1}{t+s+2}\left[(\alpha - \gamma_0) + (t+s+1)\gamma s - t(t-1)\gamma - s^2\gamma\right] \\
> \; & \frac{1}{t+s+1}\left((\alpha - \gamma_0) + (t+s)\gamma(s-1) - t(t-1)\gamma - (s-1)^2\gamma\right)
\end{aligned}
$$

which can be rewritten as

$$
\frac{1}{t+s+2}\left[-\left(\frac{1}{t+s+1}\right)((\alpha-\gamma_0) + (t+s)\gamma(s-1) - t(t-1)\gamma - (s-1)^2\gamma) + \frac{(t+s+1)}{t+s+1}(t+1)\gamma\right] > 0
$$

We have that $(\alpha - \gamma_0) - \gamma s(s-1) - t(t-1)\gamma < 0$. Hence the inequality holds if

$$
(t+s+1)(s-1) < (t+s+1)(t+1)
$$

Which is always the case whenever $s < \frac{n}{2} < t$.

(P3) We show that $\sum_{i \in N} \pi_i(g') > \sum_{i \in N} \pi_i(g)$ if $g' \in S(g,i,j)$, $n_i(g) \geq n_j(g)$ and $j \in K(g')$.

(i) $K(g') = K(g)$. We have shown the result holds when $i, j \in K^+(g)$. Suppose $j \in K_m(g)$. Then $K(g) \subseteq K(g')$ since $j \in K(g')$ and $n_k(g') > n_j(g')$ for $k \in K(g) \backslash \{j\}$. Also, $K(g') \subseteq K(g)$ since $n_l(g) = n_l(g')$ for all $l \notin K(g)$ and $\sum_{k \in K(g)} n_k(g) = \sum_{k \in K(g)} n_k(g')$.

(ii) $\pi_l(g) = \pi_l(g + ij - jk)$ for all $l \in K(g) \backslash \{i,k\}$ since $K(g) = K(g')$, $n_l(g) = n_l(g')$ and $\sum_{k \in K(g) \backslash \{l\}} n_k(g) = \sum_{k \in K(g) \backslash \{l\}} n_k(g')$.

(iii) $\pi_i(g') - \pi_i(g) = (q_i(g') - q_i(g))(q_i(g') + q_i(g)) = \gamma(q_i(g') + q_i(g))$

(iv) $\pi_j(g') - \pi_j(g) = (q_j(g') - q_j(g))(q_j(g') + q_j(g)) = -\gamma(q_j(g') + q_j(g))$

(v) $\sum_{i\in N}\pi_i(g') > \sum_{i\in N}\pi_i(g)$ since $q_i(g') > q_i(g) \geq q_j(g) > q_j(g')$.

(P4) We show that $\pi_l(g') = \pi_l(g)$ for $g' \in S(g,i,j)$ if $i,j \in K^+(g)$, $n_i(g) \geq n_j(g)$ and $l \neq i,j$.

(i) We have $K(g) = K(g')$.

The participation constraint of an agent $l \neq i,j$ is unchanged in the two networks g and g': $q_l(g) > 0 \Leftrightarrow q_l(g') > 0 \Leftrightarrow n_l(g) > \frac{\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g)-(\alpha-\gamma_0)/\gamma}{\#\{k\in N:n_k(g)\geq n_l(g)\}}$. Since $n_j(g') \geq n_l(g')$ for $l \in K_m(g)$ and $l \in K(g')$, then $j \in K(g')$.

(ii) $\pi_l(g) = \pi_l(g+ij-jk)$ for all $l \in K(g)\backslash\{i,k\}$ since $K(g) = K(g')$, $n_l(g) = n_l(g')$ and $\sum_{k\in K(g)\backslash\{l\}} n_k(g) = \sum_{k\in K(g)\backslash\{l\}} n_k(g')$.

□

Proof of Lemma 2.

(P1) Let $g \in \mathbb{G}$. Let $S \subseteq K_l(g) \subseteq K(g)$. For all $i \in S$, let $n_i(g') = n_i(g)+1$ and for all $i \notin S$, let $n_i(g') = n_i(g)$. We show that $\pi_i(g') > \pi_i(g)$ for $i \in S$.

By negative externalities, we have $K(g') \subseteq K(g)$.

(i) If $K(g) = K(g')$, then $p_i(g') = 1-(k(g)-1)/(s+\sum_{j\in K(g)}(v_i(g)+\beta)/v_j(g)) > p_i(g) = 1-(k(g)-1)/(s+\sum_{j\in K(g)} v_i(g)/v_j(g))$.

(ii) If $K(g') \subsetneq K(g)$, then let π_i be the unique Nash equilibrium payoff of agent i in a game among the agents in $K(g')$ who have the same valuation as in the network g. By negative externalities, $\pi_i > \pi_i(g)$ for all $i \in K(g')$. From step (i), we know that $\pi_i(g') > \pi_i$.

(P2) Let $g = g^T \cup g^S$, where $T \cap S = \emptyset$, $T \cup S \subsetneq N$, and $s \leq (n-1)/2 < t$. We show that if $S \subseteq K(g)$, then $\pi_i(g^T \cup g^{S\cup\{j\}}) > \pi_i(g^T \cup g^{S\cup\{j\}})$ for $i \in S$, $j \in N\backslash\{S\cup T\}$.

(i) Suppose that $K(g) = N$. Then, for $i \in S$, we have

$$\pi_i(g) = (v+(s-1)\beta)\left[1-\frac{n-1}{s+(v+(s-1)\beta)(y+\frac{1}{v})}\right]^2,$$

where $y = \frac{t}{v+(t-1)\beta} + \frac{n-t-s-1}{v}$.

(i.a) If $K(g') = N$. Then

$$\pi_i(g') = (v+s\beta)\left[1-\frac{n-1}{s+1+(v+s\beta)y}\right]^2$$

We then have that $\pi_i(g') > \pi_i(g)$ when $yv+1 > s$.

Since $K(g) = N$, we have

$$\begin{aligned} v &\geq \frac{n-2}{\frac{s}{v+(n-1)\beta}+y} > \frac{n-2}{\frac{s}{v}+y} \\ yv &> n-2-s \end{aligned}$$

This implies that $yv + 1 > s$ whenever $n - s - 1 > s$, which is satisfied when $s \leq \frac{n-1}{2}$.

(i.b) If $K(g') = T \cup S \cup \{j\} \subsetneq N$, then let π_i be the unique Nash equilibrium payoff of agent i in a game among the agents in $K(g')$ who have the same valuation as in the network g. By negative externalities, $\pi_i > \pi_i(g)$ for all $i \in K(g')$. From step (i.a), we know that $\pi_i(g') > \pi_i$.

(ii) Suppose that $K(g) = T + S$. Then, for $i \in S$ we have:

$$\pi_i(g) = (v + (s-1)\beta)\left[1 - \frac{t+s}{(v+(s-1)\beta)\left(\frac{s}{v+(s-1)\beta} + \frac{1}{\overline{v}} + \frac{t}{v+(t-1)\beta}\right)}\right]^2$$

where $\overline{v} = \frac{t+s-1}{\frac{t}{v+(t-1)\beta} + \frac{s}{v+(s-1)\beta}} > v$. We also have that

$$\pi_i(g') = (v + s\beta)\left[1 - \frac{t+s}{(v+s\beta)\left(\frac{s+1}{v+s\beta} + \frac{t}{v+(t-1)\beta}\right)}\right]^2$$

Hence, $\pi_i(g') > \pi_i(g)$ if $v < (t-1)^2\beta + st\beta$.

Since $j \notin K(g)$, we have $v < (t-1)^2\beta < (t-1)^2\beta + st\beta$.

(P3) We show that $\pi_l(g') \geq \pi_l(g)$ for $g' \in S(g,i,j)$ if $i, j \in K^+(g)$, $n_i(g) \geq n_j(g)$ and $l \neq j$.

(i) $K(g) \subseteq K(g')$ since $v_{\phi(k(g),g)} > \frac{(k(g)-1)}{k(g)} h_{k(g)}(g) > \frac{(k(g)-1)}{k(g)} h_{k(g)}(g')$, where the first inequality holds since agent $\phi(k(g), g)$ participates in the contest under network g and the second holds since the harmonic mean of a set of numbers is reduced through a mean preserving spread on these numbers.

(ii) $p_l(g) < p_l(g')$ since $(1 - \frac{(k(g)-1)h_{k(g)}(g)}{k(g)v_l(g)}) < (1 - \frac{(k(g)-1)h_{k(g)}(g')}{k(g)v_l(g)}) < (1 - \frac{(k(g')-1)h_{k(g')}(g')}{k(g')v_l(g)})$ where the first inequality holds since the harmonic mean of a set of numbers is reduced through a mean preserving spread on these numbers, and the second holds by application of Lemma 1 in Grandjean et al. (2014).

(P4) We show that $\sum_{i \in N} \pi_i(g') > \sum_{i \in N} \pi_i(g)$ if $g' \in S(g,i,j)$, $n_i(g) \geq n_j(g)$ and $j \in K(g')$. In Grandjean et al. (2014), it is shown that the sum of payoff in a given network g is given by $W(g) = \sum p_i^*(g)v_i(g) - \sum e_i^*(g)$, where $\sum p_i^*(g)v_i(g) = \sum v_i(g) - (k(g)-1)h_{k(g)}(g)$, while $\sum e_i^*(g) = \frac{(k(g)-1)}{k(g)} h_{k(g)}(g)$.

(i) $K(g) \subseteq K(g')$ since $j \in K(g')$, and $\frac{(k(g)-1)}{k(g)} h_{k(g)}(g) > \frac{(k(g)-1)}{k(g)} h_{k(g)}(g')$.

(ii) $\sum_{i \in N} e_i^*(g) > \sum_{i \in N} e_i^*(g')$

(ii.a) If $K(g) = K(g')$, the result holds since the harmonic mean of the valuation of participating agents is lower in g' than in g.

(ii.b) If $K(g) \subsetneq K(g')$, the result holds since for $j \in K(g') \backslash K(g)$, we have $\sum_{i \in N} e_i^*(g) \geq v_j > \sum_{i \in N} e_i^*(g')$.

(iii) $W(g') > W(g)$

(iii.a) If $K(g) = K(g')$, then this holds by (ii).

(iii.b) If $K(g) \subseteq K(g')$, then

$W(g') - W(g) = \sum_{j \in K(g') \setminus K(g)} v_j(g) - (k(g')+1) \sum_{i \in N} e_i^*(g') + (k(g)+1) \sum_{i \in N} e_i^*(g)$.
$= \sum_{j \in K(g') \setminus K(g)} v_j(g) - (k(g') - k(g)) \sum_{i \in N} e_i^*(g') + (k(g)+1)(\sum_{i \in N} e_i^*(g) - \sum_{i \in N} e_i^*(g'))$

For all $j \in K(g') \setminus K(g)$, we have $v_j(g') > \sum_{i \in N} e_i^*(g')$ by the participation constraint of j. Thus, $\sum_{j \in K(g') \setminus K(g)} v_j(g) - (k(g') - k(g)) \sum_{i \in N} e_i^*(g') > 0$ and the result then holds since $(\sum_{i \in N} e_i^*(g) - \sum_{i \in N} e_i^*(g')) > 0$.

□

In the proof of Propositions 2 and 3, we note by $S_1(g) = \{i \in N \mid \pi_i(g) \geq \max\{\widehat{\pi}, \overline{\pi}_1\}\}$ the set of agents whose payoff in the network g is greater than the maximal per capita payoff in a 2-clique network, and by $S_2(g) = \{i \in N \mid \pi_i(g) < \max\{\widehat{\pi}, \overline{\pi}_1\}\}$ the remaining agents. We first introduce some lemmas.

Lemma 3. *Let $\Gamma \in \mathcal{G}$ be such that $P1$ is satisfied. We have $K(\widetilde{g}^S) = S$ if $s \geq \widetilde{s}$.*

Proof. Suppose on the contrary that $K(\widetilde{g}^S) = N$. Let $K \subseteq S$ be such that $k = s - \widetilde{s}$. We have $K(\widetilde{g}^{S \setminus K}) = N$ by negative externalities and $P1$, a contradiction since $s - k = \widetilde{s}$.

□

Lemma 4. *Let $\Gamma \in \mathcal{G}$ be such that $P1$ is satisfied. Let $g \in \mathbb{G}$. We have $\pi_i(g) \leq \pi_i(g^{N(g)})$ for $i \in N^-(g)$.*

Proof. Let $g \in \mathbb{G}$. Let $i \in N^-(g)$. Let g' be such that for all $k \in N(g)$, we have $d_k(g') = d_i(g)$. We have $\pi_i(g) \leq \pi_i(g') \leq \pi_i(g^{N(g)})$, where the first inequality holds by negative externalities and the second by $P1$.

□

Lemma 5. *Let $\Gamma \in \mathcal{G}$ be such that $P1$ is satisfied. Let $\widehat{\pi} > \overline{\pi}_1$. Let $g \in \mathbb{G}$ be such that $s_1(g) \geq \widehat{s}$. Then*

(i) $n_i(g) \geq s_1(g) - 1$ for all $i \in S_1(g)$
(ii) $n_i(g) \geq s_1(g)$ for all $i \in S_1(g)$ if $s_1(g) > \widehat{s}$ or $k(g) > s_1(g)$.

Proof. Let $\widehat{\pi} > \overline{\pi}_1$. Let g be such that $s_1(g) \geq \widehat{s}$.

(i) By contradiction, suppose that $n_i(g) < s_1(g) - 1$ for some $i \in S_1(g)$. Without loss of generality, suppose $i \in \arg\min_{j \in S_1(g)} n_j(g)$. Let g' be such that $d_k(g') = d_i(g)$ for all $k \in S_1(g)$ and $d_k(g') = 0$ for all $k \in S_2(g)$. We have $\pi_i(g) \leq \pi_i(g') < \pi_i(g^{S_1(g)}) \leq \widehat{\pi}$, where the first inequality holds by negative externalities and the second by $P1$. This contradicts $i \in S_1(g)$. Thus, $n_j(g) \geq s_1(g) - 1$ for all $i \in S_1(g)$.

(ii) Suppose that $s_1(g) > \widehat{s}$ or $k(g) > s_1(g)$. By contradiction, suppose that $n_i(g) < s_1(g)$ for some $i \in S_1(g)$, which by (i) implies $n_i(g) = s_1(g) - 1$, and $i \in \arg\min_{j \in S_1(g)} n_i(g)$. We have $\pi_i(g) \leq \pi_i(g^{S_1(g)}) \leq \widehat{\pi}$. The first inequality holds by negative externalities, strictly if

$k(g) > s_1(g)$, while the second inequality holds by definition of $\widehat{\pi}$, strictly if $s_1(g) > \widehat{s}$. This contradicts $i \in S_1(g)$. Thus, $n_i(g) \geq s_1(g)$ for all $j \in S_1(g)$.

□

Proof of Proposition 2.

Let $G = \{g^S \mid s = \widehat{s}\}$.

($\Longleftarrow$) Suppose $\widehat{\pi} > \overline{\pi}_1$. We show that G satisfies internal and external stability.

Internal Stability

Let $g, g' \in G$. By contradiction, suppose $g \ll g'$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g$ to $g_K = g'$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g' \neq g \cup h$ for some $h \subseteq g^{N \setminus N(g)}$, agents from $N(g)$ modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $N(g) \cap S_k \neq \emptyset$. We have $\pi_i(g_K) \leq \pi_i(g_k) = \widehat{\pi}$ for all $i \in N(g) \cap S_k$ since $K(g_k) = K(g)$, contradicting $g \ll g'$.

External Stability

Let $g' \notin G$. Let $g_0 = g'$. In g_0 and in the successive networks, let the agents who are not participating in the current network delete their links. Let g be the network reached this way. Formally, for all $k \geq 0$, let $g_{k+1} = g_{k_{-N(g) \setminus K(g)}}$. Let $\widehat{g} = g_K$ where g_K satisfies $g_K = g_{K+1}$. By construction, $N(\widehat{g}) = K(\widehat{g})$. If $\widehat{g} \in G$, go to the final step. Otherwise, let $g'' = \widehat{g}$.

Initial step: If $n^0(g'') < n - \widehat{s}$, go to step (i); if $n - \widehat{s} \leq n^0(g'') \leq n - \widetilde{s}$, go to step (ii) ; if $n - \widetilde{s} + 1 \leq n^0(g'') \leq \widehat{s}$, go to step (iii) and if $n^0(g'') \geq \widehat{s}$ go to step (iv).

Step (i): $n^0(g'') \leq n - \widehat{s}$

(i.a) If $s_1(g'') < \widehat{s}$, then let the agents in $T \subseteq S_2(g'') \cap N(g'')$ with $t = n - \widehat{s} + 1 - n^0(g'')$ delete their links, leading to $g''' = g''_{-T}$. We have $n^0(g''') \geq n - \widehat{s} + 1$.[17] Let $g'' = g'''$, and go to the initial step.

(i.b) If $s_1(g'') \geq \widehat{s}$, then $\{i\} \leftrightarrow_{g''} N \setminus S_1(g'')$ for each $i \in S_1(g'')$ since $g'' \notin G$ by Lemma 5. An agent from $S_1(g'')$ has at least one link with an agent in $S_2(g'')$. Let $g = g''_{-S_2(g'')}$.

(i.b.1) If $s_1(g) = \widehat{s}$ and $g^{S_1(g)} \nsubseteq g$, then in g'' let the agents from $S_2(g'')$ delete their links leading to the network g. In g, let the agents in $S_2(g)$ delete their links to reach the network $g''' = g_{-S_2(g)}$. Notice that $n_i(g''') < \widehat{s} - 1$ for $i \in \arg\min_{j \in S_1(g)} n_j(g''')$ since $N(g''') \subseteq S_1(g)$ and $g^{S_1(g)} \nsubseteq g$. It follows that $i \in S_2(g''')$. Let agent i delete his links to reach $g'''' = g'''_{-i}$. Each agent who deviates in a network in the sequence from g'' to g'''' cuts all his links. There is at

[17] We have $n^0(g''') = n - \widehat{s} + 1$ if $N_i(g'') \nsubseteq T$ for all $i \in N(g'') \setminus T$. Otherwise, $n^0(g''') > n - \widehat{s} + 1$.

most $n - \widehat{s} + 1$ such agents, and there are at least $n - \widehat{s} + 1$ unconnected agents in the network reached. Let $g'' = g''''$ and go to the intitial step.

(i.b.2) If $s_1(g) = \widehat{s}$ and $g^{S_1(g)} \subseteq g$, then in g'' let the agents from $S_2(g'')$ delete their links but the link $i_1 i_2$ where $i_1 \in S_1(g)$ and $i_2 \in S_2(g'') \cap N(g'')$ leading to the network $g''' = g + i_1 i_2$. Then, let the agents from $S_2(g) \backslash \{i_2\}$ delete their links in order to reach the network $g'''' = g^{S_1(g)} + i_1 i_2$. Notice that $S_1(g'''') = \{i_1\}$. Then, let i_2 and $j \in S_1(g) \backslash \{i_1\}$ delete their links. The network reached this way is $g''''' = g^{S_1(g) \backslash \{j\}}$. Notice that $S_1(g''') \subseteq S_1(g) \cup \{i_2\}$ so that $S_2(g) \backslash \{i_2\} \subseteq S_2(g''') \backslash \{i_2\}$. Thus, the agents deleting a link in g''' have a payoff smaller than $\widehat{\pi}$. Each agent who deviates in a network in the sequence from g'' to g''''' cuts all his links. There is at most $n - \widehat{s} + 1$ such agents, and there is $n - \widehat{s} + 1$ unconnected agents in the network reached. Let $g'' = g'''''$ and go to the initial step.

(i.b.3) If $s_1(g) \neq \widehat{s}$, then in g'' let the agents from $S_2(g'')$ delete their links leading to the network g. Let $g'' = g$ and go to the intitial step.

Step (ii): $n - \widehat{s} \leq n^0(g'') \leq n - \widetilde{s}$

Let $i \in N^-(g'')$. We have $i \in S_2(g'')$ since $\pi_i(g'') \leq \pi_i(g^{N(g)}) < \widehat{\pi}$ where the first inequality holds by Lemma 4, and the second by definition of $\widehat{\pi}$. Let agent i delete all his links leading to $g''' = g''_{-i}$. Let $g'' = g'''$ and go to the initial step.

Step (iii): $n - \widetilde{s} + 1 \leq n^0(g'') < \widehat{s}$

Let the agents from $N^0(g'')$ form a component where agent $k \in N^0(g'')$ has either $n-1$ or n links while each agent in $N^0(g'') \backslash \{k\}$ has d links, where $d = \min\{n_l(g''), n^0(g'') - 1\}$ for $l \in N^-(g'')$. Let g''' be the network reached this way.

(ii.a) If $\min\{n_l(g''), n^0(g'') - 1\} = n_l(g'')$, then $\pi_l(g''') < \pi_l(g^{N \backslash \{k\}}) \leq \widehat{\pi}$. The first inequality holds by Lemma 4, and the second by definition of $\widehat{\pi}$. In g''', let $\{l\} \cup N^0(g'')$ delete their links leading to $g'''' = g''_{-l}$. Thus, $n^0(g'''') \geq n^0(g'') + 1$. Then, let $g'' = g''''$ and go to the intitial step.

(ii.b) If $\min\{n_l(g''), n^0(g'') - 1\} = n^0(g'') - 1$, let the agents from $N^0(g'')$ form a complete component. Let $i \in N^-(g'')$. We have $\pi_i(g''') \leq \pi_i(\widetilde{g}^{N(g'')}) \leq \overline{\pi}_1 < \widehat{\pi}$. The first inequality holds by negative externalities and strong degree monotonicity, and the second by definition of $\overline{\pi}_1$. In g''', let $\{i\} \cup N^0(g'')$ delete their links leading to $g'''' = g''_{-i}$. Thus, $n^0(g'''') \geq n^0(g'') + 1$. Let $g'' = g''''$ and go to the initial step.

Step (iv): $n^0(g'') \geq \widehat{s}$

Let $g^* = g''$. Let D be the set of agents who deviate in a network in the sequence from $\widehat{g}$ to g^*. By construction, $d \leq \widehat{s}$. In g^*, let the agents from D and $\widehat{s} - d$ agents from $N^0(g'') \backslash D$ form a completely connected component in order to reach g'''. In g''', the agents with less than $\widehat{s} - 1$ links do not participate. They delete their links, leading to the network $g'''' \in G$. Then go to the final step.

Final Step

The final step is either is reached directly by letting nonparticipating agents cut their links, or it is reached from step (iv). The set of unconnected agents is strictly larger after implementing the modifications of steps (i), (ii), or (iii). As a consequence, the algorithm reaches step (iv) with probability 1 if passes through the initial step. If an agent modifies the network at some point in the sequence, he is either deleting links in a network in which he is not participating, or he has a payoff stricly smaller than $\widehat{\pi}$ in the current network and is looking forward to get $\widehat{\pi}$ in the end network.

($\Rightarrow$) Suppose $\widehat{\pi} \leq \overline{\pi}_1$. We show that G does not satisfy external stability. Take $g' = \widetilde{g}^S$ such that $\#S = \overline{s}$. By contradiction, suppose $g' \ll g$ for some $g \in G$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g'$ to $g_K = g$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g' \nsubseteq g$, agents from S modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $S \cap S_k \neq \emptyset$. We have $\pi_i(g_K) = \widehat{\pi} \leq \overline{\pi}_1 \leq \pi_i(g_k)$ for all $i \in N(g) \cap S_k$ where the last inequality holds by negative externalities, contradicting $g' \ll g$.

□

Lemma 6. *Let $\Gamma \in \mathcal{G}$ be such that P1 is satisfied. Let $g \in \mathbb{G}$. Let $i \in N^-(g)$. If $n_i(g) \geq n^0(g) - 1$, let $g' = g \cup g^{N^0(g)}$. If $n_i(g) < n^0(g) - 1$, let $g' = g \cup h$ where $h \subseteq g^{N^0(g)}$ such that $n_j(g') \in \{n_i(g) - 1, n_i(g)\}$ for some $j \in N^0(g)$ while $n_k(g') = n_i(g)$ for all $k \in N^0(g) \backslash \{j\}$. Then $\pi_i(g') \leq \overline{\pi}_1$, with strict inequality if $g' \neq \widetilde{g}^S$ for $s = \overline{s}$.*

Proof. (i) Let $g \in \mathbb{G}$ be such that $n_i(g) \geq n^0(g) - 1$ for $i \in N^-(g)$. Let $g' = g \cup g^{N^0(g)}$. Then

$\pi_i(g') \leq \pi_i(g'') \leq \pi_i(\widetilde{g}^{N(g)}) \leq \overline{\pi}_1$, where $g'' = h'' \cup g^{N^0(g)}$, and $h'' \subseteq g^{N(g)}$ such that $n_j(g'') \in \{n_i(g) + 1, n_i(g)\}$ for some $j \in N(g)$ while $n_k(g'') = n_i(g)$ for all $k \in N(g) \backslash \{j\}$. The first inequality holds by negative externalities, the second by P1, and the third by definition of $\overline{\pi}_1$. If $n(g) \neq \overline{s}$, the last inequality holds strictly, while if $n(g) = \overline{s}$ and $g \neq g^{N(g)}$, then $n_i(g') < n(g) - 1$ and the second inequality holds strictly.

(ii) Let $g \in \mathbb{G}$ be such that $n_i(g) < n^0(g) - 1$ for $i \in N^-(g)$. Let $g' = g \cup h$ where $h \subseteq g^{N^0(g)}$ such that $n_j(g') \in \{n_i(g) - 1, n_i(g)\}$ for some $j \in N^0(g)$ while $n_k(g') = n_i(g)$ for all $k \in N^0(g) \backslash \{j\}$. Then $\pi_i(g') \leq \pi_i(g'') < \pi_i(g^{N \backslash \{k\}}) \leq \overline{\pi}_1$ for $k \neq i$, where g'' is such that where $g'' = h'' \cup h$, and $h'' \subseteq g^{N(g)}$ such that $n_j(g'') \in \{n_i(g) + 1, n_i(g)\}$ for some $j \in N(g)$ while $n_k(g'') = n_i(g)$ for all $k \in N(g) \backslash \{j\}$. The first inequality holds by negative externalities, the second by negative externalities and P1, and the third by definition of $\overline{\pi}_1$.

□

Lemma 7. *Let $\Gamma \in \mathcal{G}$ be such that P1 and P2 are satisfied. Let $g = g^S \cup h$ where $s = \overline{s}$ and $h \subsetneq g^{N \backslash S}$. Then $\pi_i(g) < \overline{\pi}_2$ for $i \in N^-(g)$.*

Proof. Let $g = g^S \cup h$ where $s = \overline{s}$ and $h \subsetneq g^{N \backslash S}$. Let $i \in N^-(g)$. Let $h' \subseteq g^{N(g) \backslash S}$ be such that $n_k(h') \in \{n_i(g), n_i(g)+1\}$ for some $k \in N(g) \backslash S$ and $n_j(h') = n_i(g)$ for all $j \in N(g) \backslash (S \cup k)$. Then $\pi_i(g) \leq \pi_i(g^S \cup h') \leq \pi_i(g^S \cup g^{N(g) \backslash S}) \leq \pi_i(g^S \cup g^{N \backslash S})$, where the first inequality holds by negative externalities, the second by $P1$ and the third by $P2$. Since $h \subsetneq g^{N \backslash S}$, at least one inequality holds strictly.

□

Proof of Proposition 3

Let $G = \{g \subseteq g^N \mid g = \widetilde{g}^S \text{ such that } \#S = \overline{s}\}$.

($\Longleftarrow$) Suppose $\overline{\pi}_1 > \widehat{\pi}$. We show that G satisfies internal and external stability.

Internal Stability

Let $g, g' \in G$. Let $g = \widetilde{g}^S$ with $s = \overline{s}$. By contradiction, suppose $g \ll g'$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g$ to $g_K = g'$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $N_i(g') \nsubseteq N_i(g)$ for some $i \in S$, agents from S modify the network at some point in the sequence. Let g_k be the first network in the sequence where $S \cap S_k \neq \emptyset$. We have $g_k = g^S \cup h$ where $h \subseteq g^{N \backslash S}$. Thus, $\pi_i(g_K) \leq \pi_i(g_k)$ for all $i \in S \cap S_k$ by Property 2, contradicting $g \ll g'$.

External Stability

Let $g' \notin G$. We construct a sequence of networks going from $g_0 = g'$ to $g_K = g \in G$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1} and $\pi_i(g_T) > \pi_i(g_{t-1})$ for all $i \in S_{t-1}$. Let $\gamma(g') \in \mathbb{G}$ be the unique network reached by successively deleting all the links of the agents with a payoff strictly smaller than $\overline{\pi}_1$. Formally, let $g_0 = g'$. For all $k \geq 0$, let $g_{k+1} = g_{k_{-S_2(g_k)}}$. Let $\gamma(g') = g_K$ where g_K satisfies $g_K = g_{K+1}$.

(i) If $s_1(\gamma(g')) \leq n - \overline{s}$. Let $g_0 = g'$. For all $k \geq 0$, let $g_{k+1} = g_{k_{-S_2(g_k)}}$. Let L be the smallest integer such that $s_1(g_{L+1}) \leq n - \overline{s}$. For $k = 0, 1, ..., L-1$, let the agents in $S_2(g_k)$ successively delete their links, leading to the network g_L. Let the agents from $T \subseteq S_2(g_{L+1}) \backslash S_2(g_L)$, where $t = \overline{s} - s_2(g_L)$ delete their links in g_L. Then let the agents from T and $S_2(g_L)$ form a strongly connected component, leading to the network $g'' = g^S \cup g'_{-S}$ where $S = T \cup S_2(g_L)$. Let $g' = g''$ and go to step (iii).

(ii) If $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$. For $k = 0, 1, 2, ...$, let the agents in $S_2(g_k)$ successively delete their links, leading to the network $g'' = \gamma(g')$. Then add links between agents in $N^0(g'')$ in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in N^0(g'')$ while $n_k(g''') = d$ for all $k \in N^0(g'') \backslash \{j\}$, where $d = min\{n^0(g'') - 1, n_l(g'')\}$ for $l \in N^-(g'')$. We have $\pi_i(g''') < \overline{\pi}_1$ by Lemma 6. Since $n_k(g''') \leq n_i(g''')$ for all $k \in N^0(g'')$, we have $\{i\} \cup N^0(g'') \subseteq S_2(g''')$. Let agent i and those in $N^0(g'')$ delete their links to reach $g'''' = g''_{-i}$. Let $g' = g''''$. If $s_1(\gamma(g')) \leq n - \overline{s}$, go to step (i) while if $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$, repeat step (ii).

(iii) If $s_1(\gamma(g')) = \overline{s}$, $g' = g^S \cup h$ where $s = \overline{s}$ and $h \subseteq g^{N\backslash S}$. Let $g_0 = g'$. In a network g_k, let $i \in N^-(g_k)\backslash S$ delete his links leading to $g_{k+1} = g_{k_{-i}}$. We have $\pi_i(g_k) < \overline{\pi}_2$ by Lemma 7. Let g_L be such that $g_L = g_{L+1}$. By construction, $g_L = g^S$. Then, the agents in $N\backslash S$ add each link between them leading to $g^* = g^S \cup g^{N\backslash S}$, and go to the end step (vii).

(iv) If $s_1(\gamma(g')) = \overline{s}$, $g' = g^S \cup h$ where $s = \overline{s}$ and $h \nsubseteq g^{N\backslash S}$. Then $i_1 i_2 \in g'$ for $i_1 \in S$, $i_2 \in N\backslash S$. In g' and in the successive networks, the agents with a payoff smaller than $\overline{\pi}_1$ delete their links but the link $i_1 i_2$ in order to reach $g'' = g^S + i_1 i_2$. Let $g_0 = g'$. Let $g_{k+1} = g_{k_{-S_2(g_k)}} + i_1 i_2$. Let L be such that $g_{L+1} = g_L$. By construction, $g_L = g^S + i_1 i_2$, and each agent cutting a link in the path from g' to g_L has a payoff smaller than $\overline{\pi}_1$.[18] In $g'' = g^S + i_1 i_2$, agents from $N\backslash S$ add each link between them leading to $g''' = \widetilde{g}^S + i_1 i_2$. By negative externalities, we have $N\backslash\{i_1\} \subseteq S_2(g''')$. Let the agents from $N\backslash S$ and those from $T \subseteq S_2(g''')\backslash(N\backslash S)$ where $t = 2\overline{s} - n$ delete their links, and then add each link between them, leading to the network $g'''' = g^{(N\backslash S)\cup T} \cup g'_{-(N\backslash S)\cup T}$. Let $g' = g''''$ and go to step (iii).

(v) If $s_1(\gamma(g')) = \overline{s}$ and $\gamma(g') \neq g^S$. Let $g_0 = g'$. In g_0 and in the successive networks g_k, let the agents from $S_2(g_k)$ delete their links leading to $g_{k+1} = g_{k_{-S_2(g_k)}}$. This eventually leads to the formation of the network $g'' = \gamma(g')$. Notice that $n_l(g'') < \overline{s} - 1$ for $l \in N^-(g'')$ since $g'' \neq g^S$ and $n(g'') = \overline{s}$. Then, let the agents from $S_2(g'')$ add links between them in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in S_2(g'')$ while $n_k(g''') = d$ for all $k \in S_2(g'')\backslash\{j\}$, where $d = min\{s_2(g'') - 1, n_l(g'')\}$. We have $\pi_i(g''') < \overline{\pi}_1$ by Lemma 6. Let $g' = g'''$. If $s_1(\gamma(g')) \leq n - \overline{s}$, go to step (i) while if $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$, go to step (ii).

(vi) If $s_1(\gamma(g')) > \overline{s}$. Let $g_0 = g'$. In g_0 and in the successive networks g_k, let the agents from $S_2(g_k)$ delete their links leading to $g_{k+1} = g_{k_{-S_2(g_k)}}$. This eventually leads to the formation of the network $g'' = \gamma(g')$. Let $i \in N^-(g'')$. Then, let the agents from $S_2(g'')$ add links between them in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in S_2(g'')$ while $n_k(g''') = d$ for all $k \in S_2(g'')\backslash\{j\}$, where $d = min\{s_2(g'') - 1, n_l(g'')\}$ for $l \in N^-(g'')$. We have $\pi_i(g''') < \overline{\pi}_1$ by Lemma 6.

(vi.a) If $s_1(\gamma(g''')) > \overline{s}$, let $g_0 = g'''$. In g_0 and in the successive networks g_k, let the agents from $S_2(g_k)$ delete their links leading to $g_{k+1} = g_{k_{-S_2(g_k)}}$. This eventually leads to the formation of the network $g'''' = \gamma(g''')$. Let $g' = g''''$ and repeat step (vi)

(vi.b) If $s_1(\gamma(g''')) = \overline{s}$ and $\gamma(g''') = g^S$ for $s = \overline{s}$, then $ij \in g'''$ for $j \in S$.[19] Let $g_0 = g'''$. In g_0 and in the successive networks g_k, let the agents from $S_2(g_k)$ delete their links but the link ij leading to $g_{k+1} = g_{k_{-S_2(g_k)}} + ij$. This eventually leads to the formation of the network $g'''' = g^S + ij$. Then let $g' = g''''$ and go to step (iv).

[18] Indeed, let g_k be the network in the path from g' to g_L such that $i_2 \in S_2(g_k)$. In every network $g_{k'}$ where $k' \leq k$, we have $g_{k'} = \overline{g}_{k'}$ so that $S_2(g_{k'}) = S_2(\overline{g}_{k'})$. In every network $g_{k'}$ where $k' > k$, we have $g_{k'} = \overline{g}_{k'} + i_1 i_2$. By Assumption 2, it follows that $S_2(\overline{g}_{k'}) \subseteq S_2(g_{k'})$.

[19] Such a link exists since we would otherwise have $d_i(g'') \leq n - \overline{s} - 1 - n^0(g'')$, $d_k(g'') \geq \overline{s} - 1$ for all $k \in S$ and $d_j(g'') \geq d_i(g'')$ for all $j \in S_1(g'')$. But then $\pi_i(g'') \leq \pi_i(g^S \cup g^{N\backslash(S \cup N^0(g''))}) \leq \pi_i(\widetilde{g}^S) \leq \overline{v}_2$, where the first inequality holds by negative externality and the second holds by minority returns to scale. This then contradicts $i \in S_1(g'')$.

(vi.c) If $s_1(\gamma(g'''))=\overline{s}$ and $\gamma(g''')\neq g^S$ for $s=\overline{s}$. Let $g_0=g'''$. In g_0 and in the successive networks g_k, let the agents from $S_2(g_k)$ delete their links leading to $g_{k+1}=g_{k_{-S_2(g_k)}}$. This eventually leads to the formation of the network $g''''=\gamma(g''')$. Then let $g'=g''''$ and go to step (v).

(vi.d) If $s_1(\gamma(g'''))<\overline{s}$, let $g'=g'''$ and go to step (i) if $s_1(\gamma(g'))\leq n-\overline{s}$, or to step (ii) if $n-\overline{s}<s_1(\gamma(g'))<\overline{s}$.

(vii) End Step: The algorithm describes a sequence by which the network g' is indirectly dominated by the network $g^*\in G$. With probability one, the algorithm reaches the end step (vii). In step (ii), each deviating agent has a payoff smaller than $\overline{\pi}_2$ at the network where he deviates, and get $\overline{\pi}_2$ in g^*. In the others steps, the deviating agents have striclty less than $\overline{\pi}_1$ when they modify the network, and get $\overline{\pi}_1$ in g^*. We thus have $g'\ll g^*$.

($\Rightarrow$) Suppose $\overline{\pi}_1\leq\widehat{\pi}$. We show that G does not satisfy external stability. Take $g'=g^S$ such that $\#S=\widetilde{s}$. By contradiction, suppose $g'\ll g$ for some $g\in G$. Let $g_0,g_1,...,g_K$ be a sequence of networks going from $g_0=g'$ to $g_K=g$ such that for each $t=1,2,...,K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g'\nsubseteq g$, agents from T modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $S\cap S_k\neq\emptyset$. We have $\pi_i(g_K)=\overline{\pi}_1\leq\widehat{\pi}=\pi_i(g_k)$ for all $i\in N(g)\cap S_k$ where the last equality holds since $K(g_k)=S$, contradicting $g'\ll g$.

□

Let us introduce some lemmas that are used in the proof of Proposition 5, 6, and 7. Lemma 8 shows that welfare is higher in g' than in g if g' is obtained from g through a switch among strong agents, and others among nonparticipating agents. For for $i,j\in N$ such that $n_i(g)\geq n_j(g)$, let $S^*(g,i,j)=\{g'\in\mathbb{G}\mid n_i(g')=n_i(g)+1, n_j(g')=n_j(g)-1, n_k(g')=n_k(g)$ for all $k\in K(g)\backslash\{i,j\}$, $\sum n_i(g')=\sum n_i(g)$ and $n_k(g')\leq\max_{l\in E(g)}n_l(g)$ for all $k\in E(g)\}$.

Lemma 8. *Let $g'\in S^*(g,i,j)$ for $i,j\in K^+(g)$. Then $W(g')>W(g)$.*

Proof. Let $g'\in S^*(g,i,j)$ where $i,j\in K^+(g)$. Let g'' be such that $n_i(g'')=n_i(g)$ for all $i\in K(g)$ and $n_j(g'')=n_j(g')$ for all $j\in N\backslash K(g)$. We have $W(g')>W(g'')$ by Property 3 and $W(g)=W(g'')$ since the set of participating agents and their degree is equal in the two networks.

□

Lemma 9 shows that the neighborhood of strong agents is nested.

Lemma 9. *Let $g\in\overline{G}$. Let $i,j\in K^+(g)$ with $n_i(g)\geq n_j(g)$. Then, $N_j(g)\subseteq N_i(g)$.*

Proof. Let $g\in\overline{G}$. Let $g^*=g_{-N\backslash K^+(g)}$. Let $i,j\in K^+(g)$ with $n_i(g)\geq n_j(g)$. By contradiction, suppose $N_j(g)\nsubseteq N_i(g)$. Then, there exists an agent $k\in N$ such that $jk\in g$ but $ik\notin g$. It follows that $g+ik-jk\in S(g,i,j)$, a contradiction.

□

As a consequence, if the network g is efficient, then agents in $K^+(g)$ are connected to each other through a nested-split graph. Lemma 10 shows that minor agents are connected to each agent with higher degree than their strong partners with less degree.

Lemma 10. *Let $g \in \overline{G}$. If $i_x \leftrightarrow_g K_t(g)$ for some $i_x \in K_m(g) \cup E(g)$ where $K_t(g) \subseteq K^+(g)$, then $i_x \top_g K_s(g)$ for all $s \leq t$.*

Proof. Let $g \in \overline{G}$. Let $i_x i_t \in g$ for some $i_x \in K_m(g) \cup E(g)$, and some $i_t \in K_t(g) \subseteq K^+(g)$. By contradiction, suppose $i_x i_s \notin g$ where $i_s \in K_s(g)$, $s \leq t$. Then $g - i_x i_t + i_x i_s \in S(g, i_s, i_t)$, contradicting $g \in g$.

□

Lemma 11 shows that if two participating agents are connected to nonparticipating agents, then each agent from their respective group should be connected among each other.

Lemma 11. *Let $g \in \overline{G}$. Let $i_E, j_E \in E(g)$, $i_s \in K_s(g)$ and $j_t \in K_t(g)$ with $i_s \neq j_t$. Then $i_E i_s, j_E j_t \in g \Longrightarrow K_s(g) \top_g K_t(g)$.*

Proof. Let $g \in \overline{G}$. Let $i_E, j_E \in E(g)$, $i_s \in K_s(g)$ and $j_t \in K_t(g)$ with $i_s \neq j_t$. Suppose $i_E i_s, j_E j_t \in g$ but $j_s k_t \notin g$ for some $j_s \in K_s(g)$ and $k_t \in K_t(g)$. Let $g' = g + j_s k_t - i_E i_s - j_E j_t$. If $i_s = j_s$, let $g'' = g'$. Otherwise, if $i_s \neq j_s$ then let $g'' = g' + i_s l - j_s l$ where l is such that $j_s l \in g'$ but $i_s l \notin g'$(such l exists since $n_{j_s}(g') = n_{i_s}(g') + 2$). Similarly, if $k_t = j_t$ let $g''' = g''$. Otherwise, if $k_t \neq j_t$ then let $g''' = g'' + j_t l - k_t l$ where l is such that $k_t l \in g''$ but $j_t l \notin g''$. We have $n_i(g''') = n_i(g)$ for all $i \in K(g) = K(g''')$ but $\sum_{i \in N} n_i(g) = \sum_{i \in N} n_i(g''') + 1$, contradicting $g \in g$.

□

Lemma 12 shows that if an agent j in $N \backslash K^+(g)$ is connected to some agent i in $K^+(g)$ in a network $g \in \overline{G}$, then each agent in j's group is connected to the agents with more links than i.

Lemma 12. *Let $g \in \overline{G}$. Suppose $X \leftrightarrow_g K_t(g)$ for $X \in \{K_m(g), E(g)\}$ and $t < m$, then $X \top_g K_{t'}(g)$ for all $t' < t$. If in addition, $\#K_t(g) > 1$, then $X \top_g K_t(g)$.*

Proof. Let $g \in \overline{G}$. Let $X \in \{K_m(g), E(g)\}$. By contradiction, suppose $i_x j_t \in g$ but $j_x i_s \notin g$, where $i_x, j_x \in X$, $i_s \in K_s(g)$ and $j_t \in K_t(g)$ with either $s < t < m$, or $s = t < m$ and $\#K_t(g) > 1$. By Lemma 10, $i_x j_t \in g$ implies $\{i_x\} \top_g K_u(g)$ for all $u \leq t$, and $j_x i_s \notin g$ implies $\{j_x\} \perp_g K_u(g)$ for all $u \in \{s, s+1, ..., m-1\}$. Notice that if $s = t$, $\{i_x\} \top_g K_t(g)$ while $\{j_x\} \perp_g K_t(g)$ so that we may assume $i_s \neq j_t$ without loss of generality. Agent i_x has more connections towards agents in $K^+(g)$ than agent j_x.

(i) Suppose $n_{i_x}(g) \leq n_{j_x}(g)$. It follows that agent j_x is connected to some agent k in $K_m(g)$ or in $E(g)$ to which i_x is not connected, say $j_x k \in g$ but $i_x k \notin g$ for some $k \in N \backslash K^+(g)$. Then, $g' \in S(g, i_s, j_t)$ for $g' = g + j_x i_s - i_x j_t - j_x k + i_x k$ since i_x and j_x have the same number of links in both networks while i_s has an extra link at the expense of j_t.

(ii) Suppose $n_{i_x}(g) > n_{j_x}(g)$, it follows that $X = E(g)$. Then $g' \in S^*(g, i_s, j_t)$ where $g' = g - i_x j_t + j_x i_s$, a contradiction.

□

Lemma 13 shows that the central class in a nested split graph contains more than one element.

Lemma 13. *Let g be a nested-split graph with an even number m' of classes, $N(g) = K_1(g) \cup ... \cup K_{m'}(g)$. Then, $\#K_{(m'+2)/2}(g^*) > 1$.*

Proof. Let g be a nested-split graph with an even number m' of classes, $N(g) = K_1(g) \cup ... \cup K_{m'}(g)$. By contradiction, suppose $\#K_{(m'+2)/2}(g) = 1$. Then $n_i(g) = n_j(g)$ for $i \in K_{m'/2}(g)$ and $j \in K_{(m'+2)/2}(g)$, a contradiction.

□

Lemma 14 shows that nonparticipating agents are not connected to strong agents having fewer degrees in a network $g \in \overline{G}_3$.

Lemma 14. *Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. Let $K^+(g) = K_1(g^*) \cup ... \cup K_{m'}(g^*)$. We have $E(g) \perp_g K_s(g^*)$ for all $s \geq (m'+2)/2$.*

Proof. Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. Let $K^+(g) = K_1(g^*) \cup ... \cup K_{m'}(g^*)$. By contradiction, suppose $i_E i_s \in g$ for some $i_E \in E(g)$, and $i_s \in K_s(g^*)$ where $s \geq (m'+2)/2$.

(i) Suppose n is even, then $\#K_{(m'+2)/2}(g^*) \geq 2$ by Lemma 13. By Lemma 10, we then have $\{i_E\} \top K_{(m'+2)/2}(g^*)$. But $K_{(m'+2)/2}(g^*) \perp_{g^*} K_{(m'+2)/2}(g^*)$ since g^* is a nested split graph, a violation of Lemma 11.

(ii) Suppose n is odd, then $i_E i_{s-1} \in g$ for some $i_{s-1} \in K_{s-1}(g^*)$ by Lemma 10. Since g^* is a nested split graph and $s \geq (m'+3)/2$, we have $i_{s-1} i_s \notin g^*$, a contradiction of Lemma 11.

□

Lemma 15. *Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. We have $n_i(g^*) > 0$ for all $i \in K^+(g)$*

Proof. Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. Suppose on the contrary that $n_l(g^*) = 0$ for some $l \in K^+(g)$. Notice that $\#K^+(g) \geq 2$ since $g \in g_3$. Let $i \in K^+(g)$ be such that $n_i(g) \leq n_k(g)$ for all $k \in K^+(g)$. Then, $n_i(g^*) = 0$ as we would otherwise have $g' \in S(g, l, i)$ for $g' = g - ij + lj$ where $ij \in g^*$. Let $j \in K^+(g)$, $j \neq i$. We have $N_i(g) \subseteq N_j(g)$ by Lemma 9. Then $i \perp_g E(g)$ as we would otherwise have $i_E \in N_i(g) \cap N_j(g)$ for some $i_E \in E(g)$ by Lemma 12, leading to a contradiction of Lemma 11. It follows that $i_m, j_m \in N_i(g)$ for some $i_m, j_m \in K_m(g)$ as we would otherwise have $n_i(g) \leq n_{i_m}(g)$. Also, we have $i_m k_m \notin g$ for some $j_m \in K_m(g)$ as we would otherwise have $n_i(g) < n_{i_m}(g)$. Then, let $g' = g + ij - ii_m - ij_m + i_m k_m$. Let $g'' \in S(g', i_m, j_m)$. We have $n_j(g'') = n_j(g) + 1$, $n_i(g'') = n_i(g) - 1$, and $n_k(g'') = n_k(g)$ for all $k \in N \setminus \{i, j\}$, contradicting $g \in g$.

□

Lemma 16. *Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. Then, $\#(K^+(g^*) \diagdown K_1(g^*)) \geq 2$*

Proof. Let $g \in \overline{G}_3$. Let $g^* = g_{-N \setminus K^+(g)}$. By Lemma 9, g^* is a nested split graph. By Lemma 15, $n_i(g^*) > 0$ for all $i \in K^+(g)$, so that $K^+(g^*) \top_g K_1(g^*)$. We cannot have $K^+(g^*) \diagdown K_1(g^*) = \{\emptyset\}$ since $g \in g_3$, nor $K^+(g^*) \diagdown K_1(g^*) = \{i\}$ as we would have $\{i\} \top_g K^+(g^*)$, implying $i \in K_1(g^*)$.

□

Lemma 17. *If $n_i(g) > n_j(g)$ in some network $g \in \mathbb{G}$, then $S(g, j, i) \neq \{\emptyset\}$.*

Proof. Let $g \in \mathbb{G}$ be such that $n_i(g) > n_j(g)$. Then there is $k \in N$ such that $ik \in g$ but $jk \notin g$. It follows that $g + jk - ik \in S(g, j, i)$.

□

Proof of Proposition 5

Let $g \in \overline{G}_1$. Let $K(g) = \{i_1, i_2, ..., i_{k(g)}\}$ be such that $n_{i_1}(g_{-E(g)}) \geq n_{i_2}(g_{-E(g)}) \geq ... \geq n_{i_{k(g)}}(g_{-E(g)})$.

(i) Let $n_{i_1}(g_{-E(g)}) = k(g) - 1$. Suppose on the contrary that $ij \notin g$ for $i, j \in K(g)$. Since $n_i(g) = n_j(g) = n_{i_1}(g) \geq k(g) - 1$, $ii_E, jj_E \in g$ for $i_E, j_E \in E(g)$, a contradiction of Lemma 11.

(ii) Let $n_{i_1}(g_{-E(g)}) < k(g) - 1$. Suppose on the contrary that $n_{i_1}(g_{-E(g)}) > n_{i_{k(g)-1}}(g_{-E(g)})$. Then $i_{k(g)-1} i_E$ and $i_{k(g)} j_E \in g$ for $i_E, j_E \in E(g)$, a contradiction of Lemma 11 siince we do not have $K_1(g) \top K_1(g)$.

(iii) Let $n_{i_1}(g_{-E(g)}) < k(g) - 1$. Suppose on the contrary that $n_{i_1}(g_{-E(g)}) - n_{i_{k(g)}}(g_{-E(g)}) > 1$, then $i_{k(g)} i_E$,$i_{k(g)} j_E \in g$ for $i_E, j_E \in E(g)$. Let $g' = g - i_{k(g)} i_E - i_{k(g)} j_E + i_{k(g)} i$ where $i \in K(g)$ such that $i_{k(g)} i \notin g$. Let $g'' \in S(g', i_{k(g)}, i)$. We have $n_j(g'') = n_j(g)$ for all $j \in K(g)$, but $\sum_{i \in N} n_j(g) > \sum_{i \in N} n_j(g'')$, contradicting $g \in \overline{G}$.

□

Proof of Proposition 6.

Let $g \in \overline{G}_2$. Let $g^* = g_{-N \setminus K^+(g)}$.

(i) If $X \leftrightarrow_g K_m(g)$, where $X \in \{K_m(g), E(g)\}$, then $(K_m(g) \cup X) \top_g K^+(g) \setminus \{i\}$ for some $i \in K^+(g)$.

Let $j_m j_x \in g$ for $j_m \in K_m(g)$, $j_x \in \{K_m(g), E(g)\}$. Let $i_1 \in K^+(g)$ be such that $n_{i_1}(g) \leq n_i(g)$ for all $i \in K^+(g)$, and let $i_m \in K_m(g)$ be such that $\#(N_{i_m}(g) \cap K^+(g)) \leq \#(N_i(g) \cap K^+(g))$ for all $i \in K_m(g)$.

(i.1) $K_m(g) \setminus \{i_m\} \top_g K^+(g) \setminus \{i_1\}$

Suppose on the contrary that $k_m k_1 \notin g$ for some $k_m \in K_m(g) \backslash \{i_m\}$, for some $k_1 \in K^+(g) \backslash \{i_1\}$. Then, $k_m i_1 \notin g$ by Lemma 9. Lemma 9 also implies that $N_{i_m}(g) \cap K^+(g) \subseteq N_{k_m}(g) \cap K^+(g)$. Thus, $i_m k_1, i_m i_1 \notin g$. Let $g' = g + i_m k_1 + k_m k_1 - k_1 i_1 - j_m j_x$. If $j_m = i_m$, let $g'' \in S(g', j_x, k_m)$. If $j_m = k_m$, let $g'' \in S(g', j_x, i_m)$. If $j_m \notin \{i_m, k_m\}$, let $g'' \in S(g''', j_x, k_m)$, where $g''' \in S(g', j_m, i_m)$. It follows that $g'' \in S(g, k_1, i_1)$, a contradiction.

(i.2) If $K_m(g) = \{i_m\}$, then $\{i_m\} \top_g K^+(g)$ and $E(g) \top_g K^+(g) \backslash \{i_1\}$

(i.2.a) $\{i_m\} \top_g K^+(g) \backslash \{i_1\}$.

Suppose on the contrary that $i_m j_1 \notin g$ for some $j_1 \in K^+(g)$. Then $i_m i_1 \notin g$ by Lemma 9. It follows that $\{i_1, j_1\} \perp_g E(g)$ as we would otherwise have $C^-(g) \neq \{\emptyset\}$ since $j_x \in E(g)$. Let $g' = g + i_m j_1 + j_1 j_x - i_1 j_1 - i_m j_x$. Then $g' \in S(g, j_1, i_1)$, a contradiction.

(i.2.b) $i_m i_1 \in g$

Suppose on the contrary that $i_m i_1 \notin g$. It follows that $\{i_1\} \leftrightarrow_g E(g)$, as we would otherwise have $n_{i_1}(g) \leq \kappa_1(g^*) \leq n_{i_m}(g)$. Then $C^-(g) \neq \{\emptyset\}$, a contradiction.

(i.2.c) $E(g) \top_g K^+(g) \backslash \{i_1\}$

We have $n_{i_m}(g) \geq \kappa_1(g^*) + 1$ since $\{i_m\} \top_g K^+(g)$ and $i_m j_x \in g$. It follows that $\{i_1\} \leftrightarrow_g E(g)$, say $i_1 j_E \in g$. Then, $\{j_E\} \top_g K^+(g)$ by Lemma 9. It follows that $E(g) \top_g K(g) \backslash \{i_1\}$. Suppose on the contrary that $k_E j_1 \notin g$ for some $k_E \in E(g)$, $j_1 \in K^+(g) \backslash \{i_1\}$. Then, $k_E i_1 \notin g$ by Lemma 9. Then, $n_{k_E}(g') \leq \kappa_1(g^*) \leq n_{j_E}(g)$ and $g + k_E j_1 - i_1 j_E \in S^*(g, j_1, i_1)$, a contradiction.

(i.3) Suppose $\kappa_m(g) > 1$, then $i_1 i \in g$ for some $i \in N \backslash K^+(g^*)$.

If not, then $n_{i_1}(g) = \kappa_1(g^*) - 1$, and $n_{j_m}(g) \geq \kappa_1(g^*) - 1$ for $j_m \in K_m(g)$ by (ii.1), a contradiction.

(i.4) Suppose $\kappa_m(g) > 1$ then $\{i_m\} \top_g K^+(g^*) \backslash \{i_1\}$.

On the contrary, suppose $i_m j_1 \notin g$ for $j_1 \in K^+(g^*) \backslash \{i_1\}$. Let $g' = g + i_m j_1 - i_1 i$. Then, if $i = i_m$, let $g'' = g$. Otherwise, if $i \neq i_m$, $S(g', i, i_m) \neq \{\emptyset\}$ since $n_i(g') < n_{i_m}(g')$. Let $g'' \in S(g', i, i_m)$. Then, $g'' \in S(g, j_1, i_1)$, a contradiction.

(i.5) Suppose $\kappa_m(g) > 1$ and $j_x \in E(g)$, then $E(g) \top_g K^+(g) \backslash \{i_1\}$

(i.5.a) Suppose $K^+(g) \top K_m(g)$

Suppose on the contrary that $j_E j_1 \notin g$ for some $j_1 \in K^+(g) \backslash \{i_1\}$ and $j_E \in E(g)$.

(i.5.a.1°) Suppose $K_m(g) \top_g K_m(g)$.

Then $n_{i_1}(g) > n_{i_m}(g) \geq \kappa_1(g^*) + \kappa_m(g)$ implies $i_1 k_E \in g$ for some $k_E \in E(g)$. Let $g' = g + j_E j_1 - i_1 k_E$. If $n_{j_E}(g') \leq n_{k_E}(g')$, let $g'' = g'$. Otherwise, if $n_{j_E}(g') > n_{k_E}(g')$, let $g'' \in S(g', k_E, j_E)$. We have $g'' \in S^*(g, j_1, i_1)$, a contradiction.

(i.5.a.2°) Suppose $k_m l_m \notin g$ for some $k_m, l_m \in K_m(g)$

Let $g' = g - i_1k_m + j_1j_E - j_mj_x + k_ml_m$. If $n_{j_E}(g') \leq n_{j_x}(g')$, let $g'' = g'$. Otherwise, if $n_{j_E}(g') > n_{j_x}(g')$, let $g'' = S(g', j_x, j_E)$. If $l_m = j_m$, let $g''' = g''$. Otherwise, if $l_m \neq j_m$, let $g''' \in S(g'', j_m, l_m)$. We have $g''' \in S^*(g, j_1, i_1)$, a contradiction.

(i.5.b) Suppose $i_1k_m \notin g$ for $k_m \in K_m(g)$

(i.5.b.1°) $\{i_1\} \perp_g E(g)$

Suppose on the contrary that $i_1j_E \in g$ for some $j_E \in E(g)$. Then, let $g' = g - i_1j_E - j_mj_x + i_1k_m$. If $j_m = k_m$, let $g'' = g'$. Otherwise, let $g'' \in S(g', j_m, k_m)$. We have $g'' \in C^-(g)$, a contradiction.

(i.5.b.2°) $i_1l_m \in g$ for some $l_m \in K_m(g)$

Suppose not, then $n_{j_m}(g) \geq \kappa_1(g)$ while $n_{i_1}(g) \leq \kappa_1(g)$, a contradiction.

(i.5.b.3°) $p_mr_m \notin g$ for some $p_m, r_m \in K_m(g)$

Suppose of the contrary that $K_m(g) \top K_m(g)$. Then $n_{j_m}(g) \geq \kappa_1(g) + \kappa_m(g) - 2$, while $n_{i_1}(g) \leq \kappa_1(g) + \kappa_m(g) - 2$, a contradiction.

(i.5.b.4°) $E(g) \top_g K^+(g) \backslash \{i_1\}$

By assumption, we have $j_mj_x \in g$. Given (ii.5.b.1°/2°/3°), we have $i_1l_m \in g$, $p_mr_m \notin g$. If in addition we have $j_Ej_1 \notin g$ for some $j_E \in E(g)$, $j_1 \in K^+(g) \backslash \{i_1\}$, then we could repeat the step (ii.5.a.2°).

Let $g' = g - i_1l_m + j_1j_E - j_mj_x + p_mr_m$. If $n_{j_E}(g') \leq n_{j_x}(g')$, let $g'' = g'$. Otherwise, if $n_{j_E}(g') > n_{j_x}(g')$, let $g'' = S(g', j_x, j_E)$. If $n_{p_m}(g'') = n_{r_m}(g'') = n_{l_m}(g'') = n_{j_m}(g'')$, let $g''' \in g''$. If $n_{p_m}(g'') = n_{r_m}(g'') > n_{l_m}(g'') = n_{j_m}(g'')$, let $g''' \in S(\overline{g}, l_m, r_m)$, where $\overline{g} \in S(g'', j_m, p_m)$. If $n_i(g'') > n_j(g'') = n_k(g'') > n_l(g'')$ for $\{i, j\} = \{p_m, r_m\}$ and $\{k, l\} = \{l_m, j_m\}$, let $g''' \in S(g'', l, i)$. We have $g''' \in S^*(g, j_1, i_1)$, a contradiction.

(ii) Let $K_m(g) \perp_g K_m(g) \cup E(g)$

(ii.a) $K_m(g) \top_g S_1$ and $K_m(g) \perp_g S_1$ for some $S_1 \subseteq K^+(g)$

This holds by Lemma 9 since $\#(N_{i_m}(g) \cap K^+(g)) = \#(N_{j_m}(g) \cap K^+(g))$ for all $j_m \in K_m(g)$.

(ii.b) $E(g) \perp_g N \backslash T_1$ for some $T_1 \subsetneq S_1$, $E(g) \top_g T_1 \backslash \{i\}$ for some $i \in T_1$.

(ii.b.1) $N_{i_E}(g) \subsetneq S_1$ for $i_E \in E(g)$

Let $i_E \in E(g)$. By (i), we have $E(g) \perp_g K_m(g)$ while $E(g) \perp_g E(g)$ since $C^-(g) = \{\emptyset\}$. Notice that $n_i(g) > n_j(g)$ for all $i \in S_1$ and $j \in K^+(g) \backslash S_1$ since $S(g, j, i) \neq \{\emptyset\}$. It follows that $N_j(g) \subseteq N_i(g)$ by Lemma 9. Then $i_E \notin N_j(g)$, as we would otherwise have $\{i_E\} \top_g S_1$, leading to a contradiction since $\#N_{i_E}(g) < \#S_1 = \#N_{i_m}(g)$ for some $i_m \in K_m(g)$.

(ii.b.2) if $N_{i_E}(g) \neq N_{j_E}(g)$, then $N_{i_E}(g) = N_{j_E}(g) \backslash \{i\}$ for some $i \in N_{j_E}(g)$.

Without loss of generality, let $E(g) = \{i_{E_1}, i_{E_2}, ..., i_{E_k}\}$ such that $n_{i_{E_1}}(g) \leq n_{i_{E_2}}(g) \leq ... \leq n_{i_{E_k}}(g)$.

(ii.b.2.1°) $N_{i_{E_1}}(g) \subseteq N_{i_{E_2}}(g) \subseteq ... \subseteq N_{i_{E_k}}(g)$.

Suppose on the contrary that $N_{i_{E_s}}(g) \nsubseteq N_{i_{E_{s+1}}}(g)$ for some $s \in \{1, 2, ..., k-1\}$. Then, $i_{E_s}i \in g$ while $i_{E_{s+1}}i \notin g$ for some $i \in S_1$. Since $n_{i_{E_{s+1}}}(g) \geq n_{i_{E_s}}(g)$, we then have $i_{E_s}j \notin g$ but $i_{E_{s+1}}j \in g$ for some $j \in S_1$. It follows that $S(g,i,j) \neq \{\emptyset\}$ and $S(g,j,i) \neq \{\emptyset\}$, a contradiction.

(ii.b.2.2°) $n_{i_{E_k}}(g) - n_{i_{E_1}}(g) \leq 1$

Suppose on the contrary that $n_{i_{E_k}}(g) - n_{i_{E_1}}(g) > 1$. Then $i_{E_1}i, i_{E_1}j \notin g$ for some $i, j \in N_{i_{E_k}}(g) \subseteq S_1$. It follows that $S^*(g,i,j) \neq \{\emptyset\}$ and $S^*(g,j,i) \neq \{\emptyset\}$, a contradiction.

□

Proof of Proposition 7.

Let $g \in \overline{G}_3$. Let $g^* = g_{-N\backslash K^+(g)}$. Let $K^+(g) = K_1(g^*) \cup ... \cup K_{m'}(g^*)$.

(i) $K_m(g)\top_g S_1$ and $K_m(g)\perp_g N\backslash S_1$ for some $S_1 \subsetneq K_1(g^*)$. We decompose the proof of this claim into 8 intermediate steps.

(i.a) If $K_m(\overline{g}) \leftrightarrow_{\overline{g}} K_s(g^*)$ for $s > 1$, then $K_m(\overline{g})\top_{\overline{g}} K_1(g^*)$ by Lemma 12.

(i.b) If $K_m(g)\top_g K_1(g^*)$, then $i_{m'}i_m, i_{m'}j_m \in g$ where $i_{m'} \in K_{m'}(g^*)$ and $i_m, j_m \in K_m(g)$, and we do not have $\{k_m\}\top_g K_m(g)$ for all $k_m \in K_m(g)$. Otherwise, we would have $n_{i_m}(g) \geq n_{i_{m'}}(g)$ for $i_m \in K_m(g)$ since $K_{m'}(g^*)\perp_g E(g)$ by Lemma 14.

(i.c) $K_m(g)\perp_g K_{m'}(g^*)$.

On the contrary, suppose $K_m(g) \leftrightarrow_g K_{m'}(g^*)$. Then $K_m(g)\top_g K_1(g^*)$ by Lemma 12. Let $i_{m'} \in K_{m'}(g^*)$. We have $i_{m'}i \notin g$ for some i such that $n_i(g^*) \geq n_{i_{m'}}(g^*)$ since $g \in \overline{G}_3$. By (i.a) we have $i_m i_{m'}, j_m i_{m'} \in g$ for some $i_m, j_m \in K_m(g)$, and $i_m k_m \notin g$ for some $k_m \in K_m(g)$. Let $g' = g + ii_{m'} - i_m i_{m'} - j_m i_{m'} + i_m k_m$. If $k_m = j_m$, let $g'' = g'$. Otherwise, if $k_m \neq j_m$, let $g'' \in S(g', j_m, k_m)$. We have $g'' \in S(g, i, i'_m)$, a contradiction.

(i.d) If $K_m(g) \leftrightarrow_g K_m(g)$, then $K_m(g)\top_g K_1(g^*)$.

Suppose on the contrary that $i_m j_m \in g$ but $k_m i_1 \notin g$, where $i_m, j_m, k_m \in K_m(g)$ and $i_1 \in K_1(g^*)$. By Lemma 16, $\#(K^+(g)\backslash K_1(g^*)) \geq 2$. Thus, $i_1 j, i_1 k \in g$ for some $j, k \in K^+(g)\backslash K_1(g^*)$. Without loss of generality, suppose $n_j(g) \geq n_k(g)$ so that $n_{i_1}(g) > n_j(g) \geq n_k(g)$. Since $k_m i_1 \notin g$, we have $K_m(g)\perp_g \{j, k\}$ by Lemma 12. Let $g' = g - i_m j_m - i_1 k + k_m i_1 + i_m j$. If $k_m = j_m$, let $g'' = g'$. Otherwise, if $k_m \neq j_m$, let $g'' \in S(g', j_m, k_m) \neq \{\emptyset\}$ by Lemma 17. We have $g'' \in S(g, j, k)$, a contradiction of $g \in \overline{G}$.

(i.e) $K_m(g)\perp_g E(g)$.

On the contrary suppose that $\{i_m\} \leftrightarrow_g E(g)$ where $i_m \in K_m(g)$. Let $S \subsetneq K_1(g^*)$ and $T \subseteq E(g)$ be such that $N_{i_m}(g) = S \cup T$. Notice that $s + t < \kappa_1(g^*)$. If not, then we would have $n_{i_m}(g) \geq n_{i_{m'}}(g)$ for $i_{m'} \in K_{m'}(g^*)$. We have $s < \kappa_1(g^*) - 1$ since $t > 0$. Thus there exists $i_1, j_1 \in K_1(g^*)$ such that $i_m i_1, i_m j_1 \notin g$. We have $S\top_g T$. If on the contrary, $ij \notin g$ for some

$i \in S, j \in T$, then $g - i_m j + ij - i_1 j_1 + i_m i_1 \in S(g, i, j_1)$, where $n_i(g) > n_{j_1}(g)$ since $i_m \in N_i(g)$ but $j_1 \notin N_j(g)$ by Lemma 9, a contradiction of $g \in \overline{G}$. Having $S \top_g T$ implies $t > 1$ as we would otherwise have $n_{i_m}(g) = n_{i_E}(g)$ for $i_E \in T$, a contradiction. But then, $g' \in C^-(g)$ for $g' = g - i_m i_E - i_m j_E - i_1 j_1 + i_1 i_m + j_1 i_m$, a contradiction.

(i.f) $K_{m'}(g^*) \top_g K_1(g^*)$ and $K_{m'}(g^*) \perp_g N \backslash K_1(g^*)$.

$K_{m'}(g^*) \top_g K_1(g^*)$ and $K_{m'}(g^*) \perp_g N(g^*) \backslash K_1(g^*)$ by definition of g^*. We also have $K_{m'}(g^*) \perp_g E(g)$ by Lemma 14 and $K_{m'}(g^*) \perp_g K_m(g)$ by (i.d).

(i.g) $N_{i_m}(g) \subsetneq K_1(g^*)$ for $i_m \in K_m(g)$.

Let $i_m \in K_m(g)$. From (i.a) to (i.e), we have $N_{i_m}(g) \subseteq K_1(g^*)$. The result then holds by (i.f) since $n_{i_m}(g) < n_{i_{m'}}(g)$ for $i_{m'} \in K_{m'}(g^*)$.

(i.h) $K_m(g) \top_g S_1$, where $S_1 \subsetneq K_1(g^*)$

By contradiction, suppose $i_m i_1 \in g$ but $j_m i_1 \notin g$ for $i_m, j_m \in K_m(g)$ and $i_1 \in K_1(g^*)$. Since $n_{i_m}(g) = n_{j_m}(g)$, we then have $j_m j_1 \in g$ but $i_m j_1$ for some $j_1 \in K_1(g^*)$. Thus, $S(g, j_1, i_1) \neq \{\emptyset\}$ and $S(g, j_1, i_1) \neq \{\emptyset\}$. This contradicts $g \in \overline{G}$.

(ii) $E(g) \perp_g N \backslash T_1$ for some $T_1 \subsetneq S_1$, $E(g) \top_g T_1 \backslash \{i\}$ for some $i \in T_1$.

See the proof of part (ii.b) of Proposition 6.

□