Equilibrium payoffs in a Bertrand-Edgeworth model with product differentiation

Nicolas Boccard[*] and Xavier Y. Wauthy[†]

March 7, 2005

Abstract

In this note, we consider a Bertrand-Edgeworth duopoly model in which products are differentiated "à la Hotelling". We start by assuming that only one of the two firms faces a capacity constraint. For this particular case, we characterize the equilibrium payoff of the unconstrained firm for the complete domain of capacity levels.

Keywords: Horizontal Differentiation, Bertrand-Edgeworth Competition

JEL Classification: L13

1 Introduction

The analysis of price competition in the presence of capacity constraints has been given a dramatic impulse by Kreps and Scheinkman (1983); building on Levitan and Shubik (1972), they pioneered the analysis of capacity commitment as a tool to alleviate price competition.

A key feature of the literature that followed them is a (quasi) exclusive focus on homogeneous industries. Regarding industries with differentiated products, the analysis of Bertrand-Edgeworth competition has mostly remained confined to extending the conjecture already made by Edgeworth, when he claims that "It will be readily understood that the extent of indeterminatness [resulting from the Edgeworth cycles] diminishes with the diminution of the degree of correlation between the articles" (Edgeworth (1925), p.121)". Examples are Shapley and Shubik (1969), Friedman (1988) or Benassy (1989). The main conclusion of these papers is that the presence of product differentiation enlarges the set of capacities for which the existence of a pure strategy Bertrand equilibrium exists. However, very few *positive* results exist for the case of differentiated markets where this pure strategy equilibrium does not exist. To the best of our knowledge, only three papers directly address this issue. Furth and Kovenock (1993) consider a game where prices are set sequentially and study the issue of endogenous timing. They allow for binding capacity constraints for the two firms. Cabral et al. (1998) also assume a sequential price setting in order to obtain explicit payoffs. Krishna (1989) characterizes a mixed strategy equilibrium in a pricing game between a quota constrained foreign producer and a domestic firm. Prices are set simultaneously. However, the equilibrium she characterizes is only valid within a limited range of parameters' value.[1] Clearly enough additional work is called for in order to improve our understanding of the nature of competition in the presence of capacity constraints *and* product differentiation. This is precisely the aim of the present note.

We consider a Hotelling duopoly model pricing game with fixed locations. We assume that one firm is possibly capacity constrained while the other holds an abritrarily large capacity. For this particular Bertrand-Edgeworth game, we characterize the payoff of the large capacity firm. We show that this firm either earns the Hotelling equilibrium payoffs or her minmax payoff (which depends negatively on the other's capacity). The note can thus be viewed as applying and extending the model proposed by Krishna (1989) where we characterize the domestic firm's equilibrium payoff for the whole range of quota values, i.e. also for quota levels where the equilibrium identified by Krishna does not exist. By the same token, our results extend of Levitan and Shubik (1972) to a market with differentiated products. Taking a broader perspective, our analysis takes a first step towards the characterization of firms' equilibrium payoffs in similar models where both firms are capacity constrained.

---

[1] While this is clearly not problematic in the problem considered by Krishna (1989), one needs a more general characterization in order to address stage games issues. Boccard and Wauthy (2003) also provide some related characterization.

2   The Hotelling model with Limited Production Capacities

We follow the standard assumptions of the Hotelling model. There is a continuum of consumers identified by their type x uniformly distributed in the $[0,1]$ interval. The two firms are sitting at the extremes of the market and sell an homogeneous product; the transportation cost is normalized to unity. The utility of a consumer with type x is thus $S - x - p_1$ should he buy product 1, $S - 1 + x - p_2$ should he buy product 2, and 0 if he refrains from consuming, where $S > 0$ and finite. Consumers maximize their utility given the set of prices (p_1, p_2).

The novelty we introduced is the production capacity. We consider exclusively $k_1 = 1$ and $k_2 \leq 1$. Marginal cost of production is 0 up to the capacity limit and equal to $+\infty$ otherwise. Firms maximize profits by setting (positive) prices non-cooperatively. Our equilibrium concept in Nash equilibrium, possibly in non-degenerated mixed strategies.

A monopoly receives a positive demand only if her price is lesser than the reservation value S. Next, the potential market for firm 1 consists of all types lesser than $S - p_1$ while for firm 2, it consists of all types greater than $1 - S + p_2$; they overlap only when $p_1 + p_2 \leq 2S - 1$ in which case the market is "covered". If this happens the indifferent consumer has type $\tilde{x}(p_1, p_2) \equiv \frac{1 - p_1 + p_2}{2}$. We may now characterize the demand addressed to each firm as

$$
\begin{align}
D_1(p_1, p_2) = 0 \qquad & \text{iff } p_1 \geq \min\{S, p_2 + 1\} \tag{1} \\
= S - p_1 \qquad & \text{iff } p_1 \in \left[\max\{S - 1, 2S - 1 - p_2\}; S\right] \tag{2} \\
= \tilde{x}(p_1, p_2) \qquad & \text{iff } p_1 \in \left[\max\{0, p_2 - 1\}; \min\{p_2 + 1, 2S - 1 - p_2\}\right] \tag{3} \\
= 1 \qquad & \text{iff } p_1 \leq \min\{p_2 - 1, S - 1\} \tag{4}
\end{align}
$$

and

$$
\begin{align}
D_2(p_1, p_2) = 0 \qquad & \text{iff } p_2 \geq \min\{S, p_1 + 1\} \tag{5} \\
= S - p_2 \qquad & \text{iff } p_2 \in \left[\max\{S - 1, 2S - 1 - p_1\}; S\right] \tag{6} \\
= 1 - \tilde{x}(p_1, p_2) \qquad & \text{iff } p_2 \in \left[\max\{0, p_1 - 1\}; \min\{p_1 + 1, 2S - 1 - p_1\}\right] \tag{7} \\
= 1 \qquad & \text{iff } p_2 \leq \min\{p_1 - 1, S - 1\} \tag{8}
\end{align}
$$

The domains defining these demands are illustrated by the plain lines on Figure 1 below. For the sake of simplicity, we also assume $S > 2$ to ensure that competition cannot be avoided i.e., each firm, if it were a monopoly, would want to cover the market.[2]

3   Sales Functions

Because of limited production capacities, firms' sales may differ from demands addressed to them. Whenever $D_2(.) > k_2$, firm 2 is not able to meet demand and must ration consumers. These rationed consumers may in turn report their purchase on the other product. In order to identify firms' sales under this configuration, we must specify a particular rationing rule; it determines which types of consumers are actually rationed and therefore possibly report their purchase on the other firm.

---

[2]The monopoly price is $\frac{S}{2}$ while the lower price for enjoying monopoly demand is $S - 1$. In the standard Hotelling literature, market coverage is almost invariably simply "assumed", i.e. S is arbitrarily large.

H 1 *Whenever $D_2(p_1, p_2) > k_2$, the efficient rationing rule applies.*

Under H1, firm 2 serves in priority the types exhibiting the largest utility for product 2, i.e. those belonging to $[1 - k_2; 1]$. Using the characterization of $D_2(p_1, p_2)$, we identify two critical values for p_2 such that firm 2 is capacity constrained:

$$S - p_2 \geq k2 \quad \Leftrightarrow \quad p_2 \leq S - k_2 \tag{9}$$

$$1 - \tilde{x}(p_1, p_2) \geq k_2 \quad \Leftrightarrow \quad p_2 \leq 1 - 2k_2 + p_1 \tag{10}$$

Using equations (9), (10) and the specification of $D_2(.)$, we may define the sales' function for firm 2 as

$$S_2(p_1, p_2) = 0 \qquad\qquad\qquad \text{iff } p_2 \geq \min\{S, p_1 + 1\} \tag{11}$$

$$= S - p_2 \qquad\qquad \text{iff } p_2 \in \left[\max\{S - k_2, 2S - 1 - p_1\}; S\right] \tag{12}$$

$$= 1 - \tilde{x}(.) \qquad \text{iff } p_2 \in \left[p_1 + 1 - 2k_2; \min\{2S - 1 - p_1, p_1 + 1\}\right] \tag{13}$$

$$= k_2 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \tag{14}$$

The possibility of rationing is illustrated on Figure 1 below by the addition of the grey and hatched areas. In order to characterize firm 1's sales function, we must identify the range of prices in which firm 2 is constrained while some rationed consumers report their purchase on firm 1. Under efficient rationing, we know that those rationed consumers exhibit types between $\tilde{x}(p_1, p_2)$ and $1 - k_2$. Therefore, as long as $S - (1 - k_2) - p_1 > 0$, all of the rationed consumers report their purchase on firm 1, which therefore benefits from sales equal to $1 - k_2$. When $p_1 > S - 1 + k_2$ and firm 2 is constrained, firm 1 benefits from monopoly sales $S - p_1$. Rewriting the condition $D_2(.) \geq k_2$ as $p_1 \geq p_2 + 2k_2 - 1$, we characterize the sales function for firm 1 using $D_1(.)$ as follows:

$$S_1(p_1, p_2) = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{iff } p_1 \geq S \tag{15}$$

$$= S - p_1 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \tag{16}$$

$$= 1 - k_2 \qquad\qquad \text{iff } p_1 \in \left[2k_2 - 1 + p_2; S - 1 + k_2\right] \tag{17}$$

$$= \tilde{x}(.) \qquad \text{iff } p_1 \in \left[2k_2 - 1 + p_2; \min\{p_2 + 1, 2S - 1 - p_2\}\right] \tag{18}$$

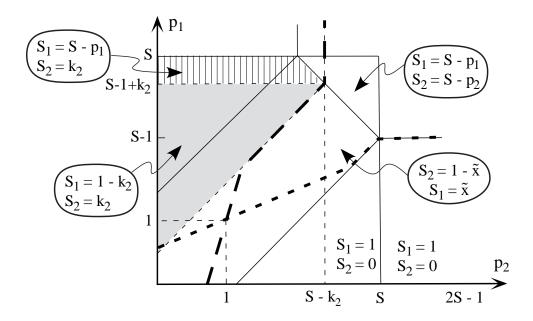$$= 1 \qquad\qquad\qquad \text{iff } p_1 \leq \min\{p_2 - 1, S - 1\} \tag{19}$$

Figure 1: Sales with one Capacity constraint

The figure contains the following labels:

$S_1 = S - p_1$, $S_2 = k_2$

$S_1 = S - p_1$, $S_2 = S - p_2$

$S_1 = 1 - k_2$, $S_2 = k_2$

$S_2 = 1 - \tilde{x}$, $S_1 = \tilde{x}$

$S_1 = 1$, $S_2 = 0$

$S_1 = 1$, $S_2 = 0$

Axes: p_1 (vertical), p_2 (horizontal). Vertical marks: S, $S-1+k_2$, $S-1$, 1. Horizontal marks: 1, $S - k_2$, S, $2S - 1$.

4   Equilibrium in the Pricing Game

We analyze the firm's best responses in games where $k_1 = 1$ and $k_2 \leq 1$. As appears from the characterization of sales functions, firms' payoffs defined by $R_i(p_1, p_2) = p_i S_i(p_1, p_2)$ for $i = 1, 2$, are continuous. Accordingly, there always exists a Nash equilibrium. We denote F_i the (possibly mixed) strategy used by firm i in a Nash equilibrium and p_i^- (resp. p_i^+) denotes the lowest (resp. highest) price named by firm i in equilibrium. We may now state our first result illustrated by the bold dashed line on Figure 1:

Lemma 1 *The best response of the capacity constrained firm (#2) is given by*

$$BR_2(p_1) = \min\left\{ S - k_2, \max\left\{ \tfrac{1+p_1}{2}, p_1 + 1 - 2k_2 \right\} \right\} \tag{20}$$

Proof: We notice that in the range where $S_2(.)$ is positive, it exhibits kinks. However, since $-\frac{\partial(S-p_2)}{\partial p_2} > -\frac{\partial(1-\tilde{x}(.))}{\partial p_2} > -\frac{\partial(k_2)}{\partial p_2}$, the sales' function is concave over the corresponding domain. As a consequence, the best response of firm 2 to any pure strategy played by firm 1 must be unique.

The candidate best response in the domain where $S_2(.) = S - p_2$ is $\frac{S}{2}$ but since we assumed $S > 2$, this optimal price is smaller than $S - 1 \leq S - k_2$ i.e., lies in the area where the capacity constraint binds. Moving to that area, the best response amounts to sell the capacity at the highest price, which is given either by $p_1 + 1 - 2k_2$ when the market is covered or by $S - k_2$ in the remaining case. Lastly, in the competition domain where $S_2(.) = 1 - \tilde{x}(.)$, the best response candidate is $\frac{1+p_1}{2}$. To obtain $BR_2(.)$, it then remains to identify the relevant best response candidate across the domain of prices, in order to obtain Lemma 1. ∎

Corollary 1 *In equilibrium, the capacity constrained firm (#2) will never play a price above $S - k_2$.*

4

Proof: Given the charaterization of $BR_2(.)$ provided in Lemma 1, it is immediate to see that the revenue of firm 2 is strictly decreasing in own price p_2 against any distribution of prices p_1 for $p_2 > S - k_2$. Therefore, $p_2 > S - k_2$ cannot be part of an equilibrium for firm 2. ∎

Lemma 2 *There exists $\underline{k}$, $\hat{p}_2$ and $\tilde{p}_2$, such that in equilibrium, the best response of the unconstrained firm (#1) is*

- *if $k_2 \geq \underline{k}$, $BR_1(p_2) = \begin{cases} S - 1 + k_2 & \text{iff} \quad p_2 \leq \hat{p}_2 \\ \min\left\{S-1, \max\left\{\frac{1+p_2}{2}, p_2 - 1\right\}\right\} & \text{iff} \quad p_2 \geq \hat{p}_2 \end{cases}$*

- *if $k_2 \leq \underline{k}$, $BR_1(p_2) = \begin{cases} S - 1 + k_2 & \text{iff} \quad p_2 \leq \tilde{p}_2 \\ \min\left\{S-1, p_2 - 1\right\} & \text{iff} \quad p_2 \geq \tilde{p}_2 \end{cases}$*

Proof: We note first that the sales function is continuous. However, it exhibits an outward kink when we pass from segment (16) to (17) in the sales function. Whenever $k_2 < 1$, the revenue function of firm 1 is equal to $p_1(1 - k_2)$ along (17) and is therefore strictly increasing in p_1 in this domain. Moreover, this segment is relevant only to the extent that $2k_2 - 1 + p_2 < S - 1 + k_2$, which is true if only $p_2 < S - k_2$. When this last condition is satisfied, the revenue $R_1(.)$ exibits a local maximum for $p_1 = S - 1 + k_2$ which precisely defines the frontier between (16) and (17). The payoff for firm 1 at this price is $\pi_1^s \equiv (1 - k_2)(S - 1 + k_2)$, a minmax value.

Along (18), $R_1(.)$ exhibits a candidate best response $\frac{1+p_2}{2}$. The corresponding best reponse candidate along (19) is given by $\min\{p_2 - 1, S - 1\}$ since $R_1(.)$ is strictly increasing along this branch. Since $R_1(.)$ is concave along (18) and (19), the best response candidate in this domain is $\min\left\{S - 1, \max\left\{\frac{1+p_2}{2}, p_2 - 1\right\}\right\}$ which is displayed by the bold dotted line on Figure 1. We check that $\frac{1+p_2}{2} > p_2 - 1$ whenever $p_2 < 3$.

In order to characterize firm 1's price best response, it remains now to compare the payoffs along $\max\left\{\frac{1+p_2}{2}, p_2 - 1\right\}$ to the minmax payoff π_1^s. Indeed, it is direct to see that the minmax payoff is dominated by $S - 1$. Firm 1's payoff along $\frac{1+p_2}{2}$ equals $\frac{(1+p_2)^2}{8}$. Solving the equation $\frac{(1+p_2)^2}{8} = \pi_1^s$ for p_2 yields

$$\hat{p}_2 \equiv -1 + \sqrt{8(1 - k_2)(S - 1 + k_2)} \tag{21}$$

Firm 1's payoff along $p_2 - 1$ is equal to $p_2 - 1$ since $S_1(.) = 1$ along this branch. Solving $p_2 - 1 = \pi_1^s$ for p_2 yields

$$\tilde{p}_2 \equiv 1 + (1 - k_2)(S - 1 + k_2) \tag{22}$$

Solving either $\hat{p}_2 \leq 3$ or $\tilde{p}_2 \geq 3$ for k_2 yields the critical capacity $\underline{k} \equiv \frac{2 - S + \sqrt{S^2 - 8}}{2}$ which determines whether the downward jump in the best reponse occurs along $\frac{1+p_2}{2}$ or $p_2 - 1$. Putting all these conditions together, we obtain the enunciated characterization of firm 1's best reponse. ∎

Notice that this best response is not continuous, exhibiting a downward jump at either $\hat{p}_2$ or $\tilde{p}_2$, depending on the value of k_2.

Corollary 2 *In equilibrium, the unconstrained firm (#1) will never play a price above $S - 1 + k_2$.*

Proof: We know that the payoff of firm 1 is strictly decreasing in the monopoly region ($D_1 = S - p_1$). Since the lower bound of this domain is given by $S - 1 + k_2$ or less, whatever the distribution of prices named by firm 2, firm 1 will never name a price above this treshold value since its payoff is strictly decreasing in this domain. ∎

As already mentioned, since firm's payoffs are continuous the existence of an equilibrium is not an issue here. However, because firm 1's best reponse is not continuous, the existence of a pure strategy equilibrium is problematic.

Lemma 3 *The unique candidate for a pure strategy equilibrium is the Hotelling equilibrium where $p_1^* = p_2^* = 1$.*

Proof: When $k_2 = 1$, it is well known that there exists a unique equilibrium $p_1^* = p_2^* = 1$.[3] Whenever $k_2 < 1$, it follows from Lemma 1 and 2 that the only other possible candidate is $(p_1, p_2) = (S - 1 + k_2, S - k_2)$. However, it is immediate to check that $\pi_1^s < S - 1 - k_2$, the payoff of firm 1 along $p_2 - 1$. Therefore, the security price $S - 1 + k_2$ is not a best reply to $S - k_2$. ∎

Lemma 4 *In equilibrium, the support of mixed strategies for both firms is bounded from below by the standard Hotelling price 1.*

Proof: If $p_1^- < 1$ then according to Lemma 1, the lowest price that firm 2 could name in this equilibrium is defined by $p_2^- \equiv \frac{1+p_1^-}{2} > p_1^-$. As a consequence, the lowest price firm 1 would name in this equilibrium, according to Lemma 2 is $\frac{1+p_2^-}{2} > \frac{1}{2}\left(1 + \frac{1+p_1^-}{2}\right) > p_1^-$, a contradiction with the definition of p_1^-. Accordingly, the lowest price named with positive probability by firm 1 must be larger than 1. Obviously, the same argument applies for firm 2. ∎

We deduce as a corollary of Lemma 1, 2 and 4 that in equilibrium, the support of firm 1's mixed strategy is included in $[1; S - 1 + k_2]$, the support of firm 2's strategy is included in $[1; S - k_2]$. Let $\bar{k} \equiv \frac{1}{2}(2 - S + \sqrt{S^2 - 2})$ be the unique root of equation $\hat{p}_2 = 1$.

Lemma 5 *Suppose $k_2 \geq \bar{k}$, then there exists a pure strategy equilibrium given by $p_1^* = p_2^* = 1$.*

Proof: Notice first that this equilibrium exists for $k_2 = 1$ since this is the standard Hotelling equilibrium. This equilibrium will continue to exist as long as the best response of firm 1 against $p_2 = 1$ is given by $\frac{1+p_2}{2}$. A necessary and sufficient condition for this is $\hat{p}_2 \leq 1 \Leftrightarrow k_2 \geq \bar{k}$. In this case indeed, the two best responses cross in the relevant domain. ∎

We are now in a position to establish the following proposition:

Proposition 3 *Suppose $k_2 < \bar{k}$, then there always exists an equilibrium in which the unconstrained firm (#1) earns its minmax payoff π_1^s.*

---

[3]See for instance Mas-Collel et al. (1995), pp 396-398

Proof: Whenever $k_2 < \bar{k}$, we have $\hat{p}_2 > 1$ so that the Hotelling equilibrium does not exist. Therefore, by Lemma 3, there is no equilibrium in pure strategy. From Lemma 1, we know that firm 2 will never name prices above $S - k_2$ in equilibrium. Let us then consider a candidate mixed strategy equilibrium.

Suppose that $p_1^+ < S - 1 + k_2$, then according to Lemma 1, in this equilibrium, the upper bound for prices named with positive probability by firm 2 is given by $\max\left\{\frac{1+p_1^+}{2}, p_1^+ + 1 - 2k_2\right\}$. This maximum is $\frac{1+p_1^+}{2}$ if $p_1^+ \leq 4k_2 - 1$; then firm 1's revenue is strictly decreasing at p_1^+ for any $p_2 \in \left[1; \frac{1+p_1^+}{2}\right]$. This contradicts the fact that p_1^+ is named with positive probability in equilibrium. Thus $p_1^+ \geq 4k_2 - 1$ must be satisfied. This implies that $p_2^+ \leq p_1^+ + 1 - 2k_2$. However, against any mixed strategy of firm 2, the payoff of firm 1 measured at p_1^+ is equal to $p_1^+(1 - k_2)$, which is strictly increasing if $p_1^+ < S - 1 + k_2$. A contradiction.

We have thus shown $p_1^+ \geq S - 1 + k_2$ and using Lemma 2, we obtain $p_1^+ = S - 1 + k_2$. Now, by Lemma 1, $p_2^+ \leq S - k_2$ so that the equilibrium payoff of firm 1 when measured at $p_1^+ = S - 1 + k_2$ must then be equal to π_1^s. ∎

References

[1] Boccard N. and X. Wauthy (2003) Optimal quotas, price competition and products' attributes, Japanese Economic Review, 54, pp. 395-408

[2] Benassy J-P. (1989), Market size and substitutability in imperfect competition: a Bertrand-Edgeworth-Chamberlin analysis, Review of Economic Studies, 56, pp. 217-234

[3] Cabral C., Kujal P. and E. Petrakis (1998), Incentives for cost reducing innovations under quantitative import restraints, Annales d'Economie et de Statistiques, 49/50, pp. 479-493

[4] Edgeworth F. (1925), The theory of pure monopoly, in Papers relating to political economy, vol 1, Mc Millan, London

[5] Friedman J. (1988), On the strategic importance of prices versus quantities, Rand Journal of Economics, 17, pp. 404-415

[6] Furth D. and D. kovenock (1993), Price leadership in a duopoly with capacity constraints and product differentiation, Journal of Economics, 57, pp. 1-35

[7] Krishna K. (1989), Trade restrictions as facilitating practices, Journal of International Economics, 26, pp. 251-270

[8] Levitan R. and M. Shubik (1972), Price duopoly and capacity constraints, International Economic Review, 13, pp. 111-123

[9] Shapley L. and M. Shubik (1969), Price strategy oligopoly with product variation, Kyklos, 22, pp. 30-44