Bertrand Edgeworth Competition in Differentiated Markets: Hotelling revisited‡

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ABSTRACT

This paper deals with situations where firms commit to capacities and compete in prices in the market for a differentiated product. First, we show that in capacity-constrained pricing games with product differentiation, there is a finite number of mixed strategies equilibria with finite support. Next, within the canonical model of Hotelling, we characterise subgame perfect equilibria (SPE) in a two-stage game with capacity commitment followed by price competition. Either equilibrium capacity choices cover the market exactly and there is no room for price competition (in which case the equilibrium outcomes replicate Cournot equilibria), or SPE involve excess capacities and limited price competition in the second period exhibiting Edgeworth cycles. These two types of SPE exist when the costs for capacity installation is negligible; however, if this cost is large enough, only SPE exhibiting Cournot outcomes survive.

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1) Introduction

Bertrand's [1883] critique of Cournot [1838] is probably one of the most famous story in undergraduate Microeconomic textbooks. According to Bertrand, two firms are enough for competitive outcomes to emerge, provided these firms set prices rather than quantities. The paradox is so unplausible from an empirical point of view that it essentially raise questions, first as to why exactly it occurs and second, about how firms manage to avoid it. Both questions have generated a huge amount of research in recent years.

On the theoretical ground the result of Bertrand rests on very fragile assumptions, namely constant returns to scale, product homogeneity and a static setting. Building on this fragility, economic scholars who found their way out of the paradox very early by relaxing one or several of these assumptions. In particular, Edgeworth [25] shows that decreasing returns to scale ensure positive profits under price competition. Hotelling [29] puts forward product differentiation in order to escape the paradox. More recently, collusive outcomes have been shown to emerge from repetition of the pricing games. All in all, models of imperfect competition avoid to fall into the Bertrand paradox by enlarging the pricing game in many directions. As such, they study the different means through which firms relax price competition.

As is widely understood nowadays, switching from a Bertrand model to a Cournot one involves more than a simple change in the strategic variable. Building on the observation that in most cases firms set prices *and* quantities, many papers tried to reconcile the two approaches. Kreps & Scheinkman [83] (KS hereafter) offers the most spectacular result in this respect. In their model, Cournot outcomes obtain as the unique subgame perfect equilibrium outcome of a stage game involving capacity commitment and price competition. In other words, the Cournot model can be viewed as the reduced form of an enlarged game in which firms ultimately do set prices under an extreme form of decreasing returns to scale. The KS result is at least as famous as it is fragile, in particular to the specification of the rationing rule (see Davidson & Deneckere [86]). Still, it points in the right direction: introducing decreasing returns to scale drives price competition towards cournotian outcomes in the sense that equilibrium market outcomes are mainly dependent on firms' output possibilities . Even though it is clear today that reconciling Bertrand and Cournot under general conditions is an hopeless task, their qualitative implications have been made somewhat compatible.

Given these results, it is quite surprising that the study of capacity-constrained pricing games remained confined to markets for homogeneous goods. After all, in almost all industries products are differentiated. The fact that product differentiation by itself relaxes price competition (and thereby avoids the Bertrand paradox) may explain why the virtues of capacity constraints have not been investigated in markets for differentiated products. Still, very little is known about the nature of price competition under decreasing returns to scale in such markets. This is clearly damaging since it means in fact that the relevance of the literature on price competition in differentiated markets is formally confined to industries exhibiting constant returns to scale.

Beyond the fact that product differentiation is not sufficient to restore the existence of pure strategy equilibria in the presence of capacity constraints (see Benassy [89] or Friedman [88]), very little is known on the nature of mixed strategy equilibria under product differentiation. Secondly, given that quantitative constraints and product differentiation are, separately, powerful in relaxing price competition, it is important to know to which extent they are substitutes or complement in this respect. Our motivation in this paper reflects these considerations. We pursue indeed two aims: first, we wish to provide a characterisation of price equilibrium in differentiated markets under capacity constraints and, second, using this result, we shall study the extent to which Cournot and Bertrand can be reconciled under product differentiation.

Several recent papers have indeed tried to reconcile Cournot and Bertrand by considering pricing games with capacity commitment while neglecting the rationing issues which where the heart of Edgeworth's argument. For instance, Dastidar [95], [97] shows that forbidding rationing is sufficient to restore pure strategy equilibria when products are homogeneous. Maggi [96] adds product differentiation to the picture and not only restores pure strategy equilibria but also ensures uniqueness. When rationing is forbidden and products are differentiated, equilibrium outcomes of a pricing game involving capacity constraints have a strong cournotian flavour.[1] It seems therefore important to know to which extent Maggi's convenient shortcut can be provided a more solid foundation than by simply assuming rationing away. To this end, it is natural to start with a "true" capacity constrained pricing game i.e., one that allows rationing.

We make two specific contributions. First, we clarify the nature of equilibria in capacity-constrained pricing games (more generally pricing games with increasing marginal costs) with product differentiation. We show that pure strategy equilibria are preserved only to the extent that quantitative constraints are loose enough. When a pure strategy equilibrium does not exist, firms use mixed strategies in equilibrium. Because of product differentiation, the equilibrium in mixed strategies has a finite support, thus involves no densities. Furthermore, there is a finite number of equilibria (in mixed strategies) and no uniqueness.

[1] Firms are "forced" to name prices in the range that corresponds to the sales of both capacities-quantities.

Thus these equilibria qualitatively differ from the ones prevailing for homogenous products where, according to the existing literature, densities and uniqueness are the rule.

Second we relate the Cournot and Bertrand results through capacity commitment in the standard Hotelling model of differentiated products. To this end, we replicate the KS analysis within the standard Hotelling model. In a subgame perfect equilibrium (hereafter SPE), capacity commitment softens price competition, as in KS, but more drastically: In most of the SPE, the capacity choices exactly cover the market and there is no room for price competition at all. Other SPE involve excess capacities and a limited price competition in the second period. Then, we show that SPE involving exact market coverage are formally equivalent to Cournot equilibria. This extends to horizontally differentiated industries the KS result according to which capacity precommitment followed by price competition leads to Cournot outcomes. It should be also mentioned that all the previously stated results are independent of the costs for capacity installation; if the capacity cost is large enough, then only SPE exhibiting Cournot outcomes exist.

The paper is organised as follows. We start in the next section by characterising the nature of a price equilibrium in a duopoly market where products are differentiated and firms face increasing marginal costs. Then we turn in section 3 to the analysis of the Hotelling model under capacity pre-commitment. We apply there the results of section 2 to the characterisation of price equilibria in the Hotelling model. In section 4 we characterise firms' capacity choices before showing how our SPE can be related to Cournot equilibrium outcomes. Section 5 concludes.

2) Equilibrium in Capacity-constrained Pricing games with Differentiated products

In order to overcome the Bertrand paradox, Edgeworth [25] shows that capacity constraints preclude the existence of pure strategy equilibria in pricing models. The argument rests on a very simple idea: a firm may benefit from spillovers when its opponent is either *not willing* or *not able* to serve full demand at prevailing prices. Indeed, the consumers who are rationed by the "low price" firm may report their purchase to the "high price" firm. When products are homogeneous, these spillovers are spectacular because a high price firm's sales jump from zero to some strictly positive level. However, only the discontinuity of the spillover is specific to the case of homogeneous goods. When raising its price against that of an opponent which sells a differentiated product, a firm will, smoothly, increase the opponent's demand up to a point where capacity becomes binding. Beyond that point, spillovers accrue, smoothly, to the "high price" firm, as in the homogeneous products case.

Therefore, the reason why an equilibrium fails to exist in the analysis of Edgeworth is still present under product differentiation.

Note that Edgeworth himself does not restrict the possibility of "cycles" to the case of homogeneous goods. In his own words, "It will be readily understood that the extent of indeterminatness diminishes with the diminution of the degree of correlation between the articles" (Edgeworth [25], p.121). On the other hand, Hotelling [29] thought that product differentiation would solve the Edgeworth problem of cycles completely, as he wrote "The assumption, implicit in their work [Cournot, Amoroso and Edgeworth] that all buyers deal with the cheapest seller leads to a type of instability which disappears when the quantity sold is considered as a continuous function of the differences in prices" (Hotelling [29] p 471, bracket added). Although Hotelling was right in arguing that continuous demand would solve the Bertrand paradox, he was wrong on the Edgeworth's front. Shapley & Shubik [69] and McLeod [85] provide a formal treatment of the role of "correlation" in Edgeworth's intuition: product differentiation is not sufficient to restore the existence of a pure strategy equilibrium in a pricing game with increasing marginal costs because profits functions typically remain non quasi-concave. However, it is of some help in the sense that product differentiation tends to enlarge the set of capacity levels for which a pure strategy equilibrium is preserved.[2]

If the non-existence of pure strategy equilibria in the presence of capacity constraints, even under product differentiation, is a (fairly) well-documented issue, very little is known about the nature of a mixed strategy equilibrium in such settings. Noticeable exceptions are Krishna (1989) and Furth and Kovenock (1992) who provide some partial characterisations. Accordingly, our first task will consist in clarifying the nature of price equilibria when both decreasing returns to scale and product differentiation are present.

To this end, we consider the market for a differentiated product. The demand addressed to firm 1 is $D(p_1,p_2)$ while that of firm 2 is the symmetric $D(p_2,p_1)$. The function $D(p_1,p_2)$ is assumed continuously differentiable of order 2 and satisfies the following assumptions:

A1) $D(0,.) > 0$

A2) $-\frac{\partial D(p_1,p_2)}{\partial p_1} \geq \frac{\partial D(p_1,p_2)}{\partial p_2} > 0$

A1) assumes that a firm's demand is positive when its price is zero whereas A2) means that own price effect on a firm's demand dominate crossed ones. Consider a complete information stage game Γ. At stage one, firms invests into technologies yielding increasing

[2]This point is studied by Friedman [88], Benassy [89], Canoy [96] and Wauthy [96]. These papers share the idea that the more differentiated the products, the more likely a pure strategy equilibrium will exist.

and convex cost functions C_1 and C_2. At stage two, they set prices. Lastly in stage three, firms perform rationing whenever they wish to and some of the rationed consumers turn to the other firm.

Before considering this last stage it is useful to describe the case where rationing is forbidden i.e., *Bertrand* competition.[3] The profit function is then $\Pi_i^B(p_i,p_j) \equiv p_i D(p_i,p_j) - C_i\big(D(p_i,p_j)\big)$. In order to guarantee uniqueness of the Bertrand equilibrium irrespective of the cost functions chosen at stage one we assume the following contraction property:

A3) $0 \le \frac{\partial^2 D}{\partial p_i \partial p_j} p_i + \frac{\partial D}{\partial p_j} \le -2\frac{\partial D}{\partial p_i} - \frac{\partial^2 D}{\partial p_i^2} p_i$

Let then $\varphi_i(p_i) \equiv \arg\max_{q \ge 0}\{qp_i - C_i(q)\}$ be the competitive supply of firm **i** (it is equal to $C_i'^{-1}(p_i)$ if C_i is strictly convex) and $\Pi_i^C(p_i) \equiv p_i\varphi_i(p_i) - C_i\big(\varphi_i(p_i)\big)$ be the *competitive* profit. It is increasing convex since $\dot{\Pi}_i^C(p_i) = \varphi_i(p_i)$ and $\varphi_i(p_i)$ is itself weakly increasing. Let us then allow firms to ration consumers whenever it is profitable for them to do so. If (p_i,p_j) is such that $D(p_j,p_i) > \varphi_j(p_j)$ then in stage three, firm **j** rations some consumers and firm **i** obtains a fraction $\lambda(p_j,p_i)$ of the residual demand, this is the spillover effect. The importance of the spillover depends on the degree of differentiation of the products, the preferences of the consumers and the rationing rule used by firms; we assume that it is continuously differentiable and satisfies:

A4) Positive spillovers decreasing with respect to own price: $\lambda(p_i,p_j) > 0$, $\frac{\partial \lambda}{\partial p_i} \le 0$ and $2\frac{\partial \lambda}{\partial p_i} + p_i \frac{\partial^2 \lambda}{\partial p_i^2} \le 0$

Given these four assumptions we analyse a price subgame $\Gamma(C_1,C_2)$ and look for subgame perfect equilibria (SPE). Note that when rationing is forbidden (no third stage in Γ) the *Bertrand* payoff applies over the whole range of prices. If rationing is permitted, a *Bertrand-Edgeworth* analysis is called for because firms' sales may differ from demands. It is well-known that in such circumstances a pure strategy equilibrium often fails to exist (see for instance Benassy [89]). The following theorem provides a general result about the nature of the equilibrium mixed strategies for such games.

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[3] Maggi (1996) provides a recent case where such a view of price competition under capacity constraints is endorsed.

THEOREM 1

Consider any price subgame $\Gamma(C_1,C_2)$ that satisfies A1-A4. Under Bertrand competition, there exists a unique pure strategy SPE. Under Bertrand-Edgeworth competition, a SPE always exists, the support of a mixed strategy price SPE is finite and prices are larger than those of the Bertrand equilibrium.

The detailed proof has been relegated to the appendix, however the intuitive argument is relatively easy to summarise. When rationing is not allowed, **A1** and **A2** ensure that a firm's payoff is concave. Assumption **A3** enables to derive the Bertrand equilibrium as the unique fixed point of the best reply operator. To show that this Bertrand equilibrium is also a lower bound to prices played in equilibrium of the Bertrand-Edgeworth competition we use the fact that Π_i^B and Π_i^C are both increasing over the domain where firm **i** wishes to ration. Then, in order to prove that firms do not use densities in equilibrium, one shows that, if firm **i** uses a density around some price p_i, then firm **j** must using a density around the price p_j that makes it willing to ration i.e., such that $D(p_j,p_i)=\varphi_j(p_j)$. Symmetry then implies that firm **i** uses a density around $\hat{p}_i$ such that $D(\hat{p}_i,p_j)=\varphi_i(\hat{p}_i)$. This process leads to lower and lower prices precisely because goods are differentiated. We reach a contradiction because firms do not put mass below the Bertrand prices in equilibrium. If firms do not use densities then support of equilibrium distributions must be finite. The equilibria characterised in Theorem 1 are quite different from those prevailing in market for homogeneous goods where firms use densities under standard assumptions on demand. Note also that the argument developed above does not help to prove uniqueness and indeed multiplicity of equilibria often obtains. This will in particular be the case for the model of capacity pre-commitment in the Hotelling market we consider hereafter.

3) Price equilibrium in the Hotelling model with capacity commitment.

In what follows, we adapt the stage-game proposed in Kreps & Scheinkman [83] to the Hotelling model of differentiation: Firms choose capacity levels and then compete in price in a horizontally differentiated market. After presenting a simplified version of the Hotelling model, we define the full game as well as the assumptions under which our analysis is conducted. Then we characterise equilibria in the pricing games. The analysis of pricing games is rather long and involved. We have chosen to concentrate on intuitive arguments in the core of the paper. Almost all technical proofs have been relegated to the appendix.

3.1) THE SET UP

Two shops are located at the boundaries[4] of the [0;1] segment along which consumers are uniformly distributed. Each consumer is identified by its address **x** in [0;1] and has a common reservation price **S**. An agent buys at most one unit of the good and bears a transportation cost, which is linear in the distance to the shop. Without loss of generality, we set the transportation cost between the two shops to 1. Therefore, the utility derived by a consumer located at **x** is thus $S - x - p_1$ if he buys the product at firm 1 (located in 0) and $S - (1 - x) - p_2$ if he buys at firm 2 (located in 1). Refraining from consuming any of the two products yields a nil level of utility[5]. Firms name prices non-cooperatively.

The essence of the Hotelling model is best summarised as follows. When firms name low prices, the market is covered (i.e. all consumers purchase one of the good). Firms' market shares are defined by the address of the *indifferent* consumer denoted by $\tilde{x}(.)$. By definition, it is the solution of $S - x - p_1 = S - (1-x) - p_2$ i.e., $\tilde{x}(p_1, p_2) \equiv \frac{1-p_1+p_2}{2}$. Consumers in $[0, \tilde{x}(p_1, p_2)]$ buy at firm 1 whose demand is $\tilde{x}(p_1, p_2)$ as consumers are uniformly distributed on [0;1]. Demand addressed to firm 2 is $1 - \tilde{x}(p_1, p_2)$. If prices are too large the market is not covered. In such cases, firm[6] **i** is a local monopoly; its demand is $\min\{1, S - p_i\}$. This happens if $S - \tilde{x}(p_i, p_j) - p_i < 0 \Leftrightarrow p_i > 2S - 1 - p_j$. The demand function of firm **i** is thus defined by equation

$$D_i(p_i, p_j) = \begin{cases} \frac{1-p_1+p_2}{2} & \text{if } p_i \leq 2S - 1 - p_j \\ \min\{1, S - p_i\} & \text{if } p_i > 2S - 1 - p_j \end{cases}.$$

Plugging this demand in the profit function, we may identify the respective argument maximisers $H(p_j) \equiv \frac{1+p_j}{2}$ for the first branch and the monopoly price $p^m \equiv \min\left\{S - 1, \frac{S}{2}\right\}$ for the second. Against a low p_j firm **i** plays $H(p_j)$ while against a large p_j it plays p^m; for prices in the middle range, the optimal response is to cover the market with $2S - 1 - p_j$.

Since $D_i(p_i,p_j)$ is piecewise linear and decreasing in p_j, profit is concave in p_j, thus the best reply to a mixed strategy is the best reply to its expectation i.e., a pure strategy. Therefore, a Nash equilibrium of this pricing game is pure. The best reply intersect at the unit price for both firms whenever $S > 3/2$. In this case, firms face no capacity constraints the unique Nash equilibrium of the pricing game is (1,1) and the market is covered. Otherwise

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[4] We choose maximum differentiation to relax price competition as much as possible. If firms find it profitable to further relax price competition through capacity precommitment, it is likely that they would face even greater incentives if they were less differentiated on the horizontal dimension.

[5] In Hotelling's original model, this possibility is not considered, formally, this correspond to an infinite S.

[6] In the remainder of the text, **i** stands for either of the firms and **j** for its competitor.

when S < 3/2, there is a continuum of equilibria on the frontier ($p_i + p_j = 2S + 1$) which entail no "real" price competition.

In order to introduce capacity commitment, we now add a preliminary step where firms choose simultaneously sales capacity k_1 and k_2 before they simultaneously choose prices p_1 and p_2 knowing the chosen capacity of their competitor. The unit cost of capacity is assumed equal to $\varepsilon > 0$. This two stage-game of complete information is denoted G. The subgame after the choices of k_1 and k_2 is denoted $G(k_1,k_2)$ and called the pricing game. Formally, G is thus identical to the game considered by KS.

Given that firms may act as local monopolists provided prices are high enough, game G is really interesting if only firms are lead to choose capacities whose sum exceeds the market size. Only in this case will they enter into a price competition at the second stage. Obviously this cannot happen for very large capacity costs. Proposition 1 clarifies this point.

PROPOSITION 1

If the unit cost of capacity installation ε is larger than $S - 1$, the unique SPE entails monopoly pricing by both firms. If $\varepsilon < S - 1$, the market must be covered in a SPE.

Proof Suppose that firm **i** is a monopoly over the market and has installed a capacity k_i, its demand is $f_i \equiv \min\{k_i, S - p_i\}$. The second period profit $p_i(S - p_i)$ is maximum for $p_i = \min\{S - k_i, \frac{S}{2}\}$, thus the first period profit is $k_i(S - k_i - \varepsilon)$ if $k_i < \frac{S}{2}$ and $\frac{S^2}{4} - \varepsilon k_i$ if $k_i \geq \frac{S}{2}$. Since the latter is decreasing with k_i, only the first matters for the optimal capacity choice which is $k^m \equiv \min\{1, \frac{S-\varepsilon}{2}\}$. Now, it clear that $\varepsilon > S - 1$ implies $k^m < 1/2$; Being located respectively at 0 and 1, both firms are able to achieve their equilibrium monopoly profit without interacting which means that k^m is a dominant strategy and thus characterise the unique SPE allocation. It is clear that when $\varepsilon < S - 1$, capacity choices such that $k_1 + k_2 < 1$ are not stable since one of the firm (may be both) has an incentive to choose at least a capacity complementary to its competitor. Yet, exact market coverage capacities ($k_1 + k_2 = 1$) subject to the constraint $\max\{k_1, k_2\} \leq k^m$ are candidates SPE of the whole game. ♦

In the sequel of the paper, we concentrate on cases where unit capacity costs are negligible relative to S. The presence of limited capacities affects firms' incentives in the pricing game in two ways. First, a limited capacity may decrease the incentive of a firm to reply to the other's price with a low price since the market share a firm is willing to serve cannot exceed its installed capacity. A second observation induced by limited capacities is that some consumers might be rationed at the prevailing prices. This possibility is the cornerstone of the price competition analysis as it may reverse firms' incentives in the price game. More precisely, one firm could find it profitable to quote a high price, anticipating the fact that some consumers will be rationed by the other firm and could therefore be willing to report their purchase to it. This was the original intuition of Edgeworth. The incentives for that behaviour basically depend on the willingness of consumers to switch to the high price firm in case of rationing. In a model with unit demand and heterogeneous consumers the extent to which rationed consumers will be recovered by this firm directly depends on who the rationed consumers are.

We follow KS in assuming that the efficient rationing rule is at work in the market. Under this rule, rationed consumers are those exhibiting the lowest reservation price for the good. Consider the example depicted on Figure 1 below.

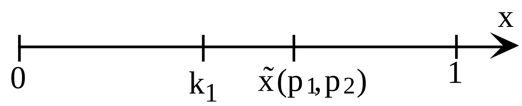

Figure 1

All consumers located in $\left[0;\tilde{x}(p_1,p_2)\right]$ want to buy at firm 1 but some will be rationed. Under efficient rationing, those are located in $\left[k_1;\tilde{x}(p_1,p_2)\right]$ and are precisely the most inclined to switch to firm 2. Despite a potentially low demand for firm 2, the fact that firm 1 is constrained by its capacity k_1, can give firm 2 an effective demand of $1 - k_1$. It is the case if $p_2 \leq S - 1 + k_1$ which is the net reservation price of the consumer located in k_1. This feature of the market allocation rule also lowers firm 2's incentives to enter a price competition "à la Bertrand" since its demand is locally independent of its own price. Note finally that in the Hotelling model with maximal differentiation, efficient rationing defines the largest residual demand for firm 2, so that, contrarily to KS, the incentives to use rationing strategically are maximised. This phenomenon will have a strong feedback on the choices of capacities. The next section studies the pricing game when the choice of capacities exceed the market size.

3.2) EQUILIBRIA IN PRICE SUBGAMES

As a first step we derive the effective sales of the firms in the pricing game using consumer demands and the rationing rule. Then, we characterise the best reply functions and identify the support of the equilibrium mixed strategies. Three types of equilibria are characterised: the pure strategy equilibrium that prevails under standard Hotelling competition, equilibria in which one firm mixes over two prices whereas the other plays a pure strategy (we refer to these as semi-mixed equilibria) and last, equilibria where both firms use non-degenerated mixed strategies. In all of the possible pricing games $G(k_1,k_2)$, at least one of these equilibria prevails but multiple equilibria often prevail. In the mixed strategy equilibria, firms use only atoms.

The shape of firms' sales can be informally captured by referring to Figure 2.

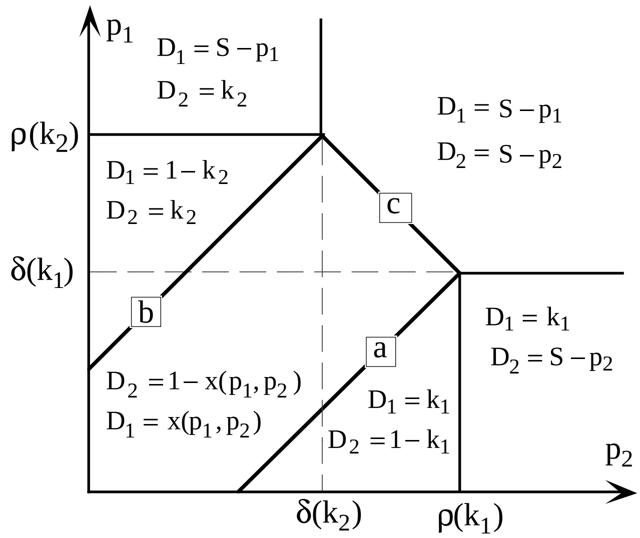

Figure 2

Assume firms quote similar (low) prices corresponding to a point in the area delimited by lines a, b and c. In this region, the classical Hotelling demands prevail. Now, if p_1 increases, firm 1 looses sales until firm 2 is constrained by her capacity i.e., we reach the upper triangle. From that point on, if p_1 increases further, D_1 remains constant at $1 - k_2$ until p_1 is so large that the market is not covered anymore. From then on, firm 1 moves along its monopoly demand function. When both prices are high, the market is not covered and, obviously, no firm is capacity constrained.

We derive now each firm's sales function formally. Since the rationing rule that we use is the efficient one, if demand addressed to firm **i** exceeds k_i, it serves the segment $[0;k_i]$. Thus, if we let D_i be the sales of firm **i**, the demand addressed to firm **j** is bounded by $1 - D_i$. On the other hand, D_i is bounded by the capacity k_i and by the monopoly sales $S - p_i$. Letting $f_i \equiv \min\{k_i, S - p_i\}$, we thus have

$$D_i \equiv \min\left\{S - p_i, k_i, 1 - D_j\right\} = \min\left\{f_i, 1 - D_j\right\}$$

Observe that there is an equivalence between $\{\tilde{x}(p_1,p_2) < f_1, 1 - \tilde{x}(p_1,p_2) < f_2\}$ and $\{D_1 = \tilde{x}(p_1,p_2), D_2 = 1 - \tilde{x}(p_1,p_2)\}$ because demands addressed to firms can be served by both while the reverse implication is true as one can see from the definition of D_i. Let us denote by (E1) the equation $\tilde{x}(p_1,p_2) < f_1$ and by (E2) the equation $1 - \tilde{x}(p_1,p_2) < f_2$. We investigate when they hold:

- $p_1 \le S - k_1 \Rightarrow f_1 = k_1$ and (E1) becomes $\tilde{x} = \frac{1 - p_1 + p_2}{2} \le k_1 \Rightarrow p_1 \ge a(k_1, p_2) \equiv p_2 + 1 - 2k_1$

- $p_1 > S - k_1 \Rightarrow f_1 = S - p_1$ and (E1) reads $\tilde{x} = \frac{1-p_1+p_2}{2} \leq S - p_1 \Rightarrow p_1 \leq c(p_2) \equiv 2S - p_2 - 1$

- $p_2 \leq S - k_2 \Rightarrow f_2 = k_2$ and (E2) reads $1 - \tilde{x} = \frac{1-p_2+p_1}{2} \leq k_2 \Rightarrow p_1 \leq b(k_2, p_2) \equiv p_2 - 1 + 2k_2$

- $p_2 > S - k_2 \Rightarrow f_2 = S - p_2$ and (E2) becomes $1 - \tilde{x} = \frac{1-p_2+p_1}{2} \leq S - p_2 \Rightarrow p_1 \leq c(p_2)$

Thus, the classical Hotelling demands $D_1 = \tilde{x}(p_1, p_2)$ and $D_2 = 1 - \tilde{x}(p_1, p_2)$ prevail if $a(k_1, p_2) \leq p_1 \leq \min\{b(k_2, p_2), c(p_2)\}$. The maximum price compatible with sales of k_i is $\delta(k_i) \equiv S - k_i$ while the price guaranteeing firm **i** a demand of $1 - k_j$ is $\rho(k_j) \equiv S - 1 + k_j$. We call this price the "security" price of firm i.

We choose $k_1 > k_2$ without loss of generality so that $k_1 > \frac{1}{2}$. As shown on Figure 2 above, $k_1 + k_2 > 1$ implies $a(k_1,.) < b(k_2,.)$ and $\delta(k_j) < \rho(k_i)$. Lastly, $a(k_1,.) = c(.) = \delta(k_1)$ for $p_2 = \rho(k_1)$ and $b(k_2,.) = c(.) = \rho(k_2)$ for $p_2 = \delta(k_2)$. The area delimited by the functions **a**, **b** and **c** will be called "the band". Observe that $k_1 > \frac{1}{2}$ implies $a(k_1,0) < 0$ while $b(k_2,0)$ can be positive (as on Figure 2) or negative.

We now derive the best reply of firm 1 to a price charged by firm 2. We already noted that $S > 1 + \varepsilon$ was a necessary condition for firm to engage into a meaningful competition. We go a step further[7] by assuming $S > 2$ to create a fierce price competition between the duopolists. Under this assumption a monopolist located at one end of the market would choose to cover the market. Technically, it implies $\frac{S}{2} < \delta(k_i)$ and $\frac{S}{2} < \rho(k_i)$ for **i** = 1,2.

LEMMA 1

*Firm **i** =1,2 never charge prices above $\rho(k_j)$; the best replies are discontinuous and defined by*

$$BR_i(p_j) = \begin{cases} \rho(k_j) & \text{if} \quad p_j \leq \gamma(k_i, k_j) \\ \frac{1+p_j}{2} & \text{if} \quad \gamma(k_i, k_j) < p_j < \alpha(k_i) \\ p_j + 1 - 2k_i & \text{if} \quad \max\{\alpha(k_i), \gamma(k_i, k_j)\} < p_j \end{cases}$$

<u>Proof</u> Let F_1 be the cumulative distribution function of the mixed strategy used by firm **1** in equilibrium. Recall that $\forall p_2, \forall p_1 \geq \rho(k_2), D_1(p_1, p_2) = S - p_1$. Since the monopoly price $\frac{S}{2}$ is less than $\rho(k_2)$, $\Pi_1(p_2,.)$ must be decreasing over $[\rho(k_2);+\infty[$ and the same is true for $\Pi_1(F_2,.) = \int \Pi_1(p_2,.)dF_2(p_2)$. Therefore F_1 ,being a best reply to F_2, has no mass above $\rho(k_2)$ and symmetrically the support of F_2 is included in $[0;\rho(k_1)]$.

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[7] A close look at the proofs shows that it is unnecessary but it simplifies the exposition by removing subcases. Notice that in the standard analysis of the Hotelling model, it is generally assumed that S *large enough* to ensure market-covering in equilibrium.

We can now restrict the study of the best reply of firm 1 to a price p_2 lesser than $\rho(k_1)$. Firm 1 can act in a classical fashion "à la Hotelling" with an aggressive price in order to gain market shares. It can also hide behind firm 2's capacity by serving that part of market that is out of reach for firm 2 i.e., the $[0;1-k_2]$ interval. Over this residual market, firm 1's payoff is given by its monopoly payoff function (this is the key feature of the Hotelling framework) and the optimal price is $\rho(k_2)$; we call it the security strategy.

Then, we distinguish four areas on Figure 2 above: the band, the lower triangle, the upper triangle and the domain of monopoly demand above $\rho(k_2)$ and **c**. For the latter we have just shown that the best choice is the lower frontier of the domain i.e., $\rho(k_2)$ for $p_2 < \delta(k_2)$ and $c(p_2)$ beyond. Observe that $c(p_2)$ is itself dominated by the best reply within the band. In the upper and lower triangles demand is constant so that profit is increasing and the best choices are respectively $\rho(k_2)$ and $a(k_1,p_2)$. This latter value is dominated by the optimum within the band. In the band the best reply is either H(.) or one of the frontiers $a(k_1,.)$ and $b(k_2,.)$. Solving for $b(k_2,p_2) > H(p_2) > a(k_1,p_2)$, the first inequality leads to $p_2 > \beta(k_2) \equiv 3-4k_2$ (it is always satisfied if $k_2 > 3/4$) while the second leads to $p_2 < \alpha(k_1) \equiv 4k_1 - 1$. Since $b(k_2,p_2)$ also belongs to the upper triangle, it is dominated by $\rho(k_2)$. This observation enables without loss of generality to take $\mathrm{Max}\{H(.), a(k_1,.)\}$ as be the best choice in the band because we will then choose between this candidate and the security strategy $\rho(k_2)$.

Summing up, the best reply is either $\varphi_1(p_2) = \begin{cases} \frac{S}{2} & \text{if } p_2 < \alpha(k_1) \\ p_2 + 1 - 2k_1 & \text{if } \alpha(k_1) < p_2 \end{cases}$ or the security price $\rho(k_2)$.

Comparing the associated profits, we derive a cut-off price $\gamma(k_1,k_2)$ such that when the price p_2 is low, firm 1 replies with a high price to benefit from the resulting rationing at firm 2. Against a high price p_2, firm 1 fights for market shares. The derivation of $\gamma(k_1,k_2)$ can be found in Lemma A.1 of the appendix. Formally, we obtain:

$$BR_1(p_2) = \begin{cases} \rho(k_2) & \text{if} \quad p_2 \le \gamma(k_1,k_2) \\ \varphi_1(p_2) & \text{if} \quad p_2 > \gamma(k_1,k_2) \end{cases}. \ \blacklozenge$$

The previous analysis enables us to define the support of an equilibrium mixed strategy.

PROPOSITION 2

In equilibrium, the support of the mixed strategy F_i is included in

$$\left[max\left\{ 1, \frac{(1-k_j)(S-1+k_j)}{k_i} \right\}; \rho(k_j) \right]$$

Proof As a corollary of Theorem 1 each firm names prices larger than the Hotelling unit price. Now observe that Firm 1 can guarantee itself the demand $1 - k_2$ by playing $\rho(k_2)$, thus its equilibrium profit is larger than $\Pi_1^s \equiv (1-k_2)(S-1+k_2)$. Now, as sales are bounded by the capacity k_1, firm 1 must name a price above $\underline{p}_1 \equiv \frac{\Pi_1^s}{k_1}$ in equilibrium. A symmetrical result holds for firm 2. The statement on the upper bound was proved in lemma 1. ♦

We now characterise the equilibria of the price subgame in proposition 3 and provide a sketch of the proofs. Analytical developments have been relegated to the appendix.

PROPOSITION 3

Three non exclusive types of price equilibria exist:
- *A) both firms quote the pure strategy Hotelling price;*
- *B) one firm plays a pure strategy and the other mixes over two atoms;*
- *C) both firms use a mixed strategy involving the same number of atoms.*

Sketch of the Proof:

A) Equilibria involving only pure strategies

If both capacities are arbitrarily close to 1, the standard Hotelling equilibrium of proposition 1 is preserved. Indeed if $\gamma(k_1,k_2) < 1$ and $\gamma(k_2,k_1) < 1$, then the best reply curves intersects at (1,1) meaning that the pure strategy price equilibrium (1,1) exists. Those inequalities define two sets A_1 and A_2 in the capacity space which are symmetric with respect to $k_2=k_1$. Their intersection is a square area $[\Phi;1]\times[\Phi;1]$ where $\Phi \equiv \frac{1}{2}\left(2-S+\sqrt{S^2-2}\right)$.

B) Equilibria involving a pure strategy and a mixed one

The set $A_2 \backslash A_1$ corresponds to a "large" k_1 and a "smaller" k_2; the pure strategy equilibrium ceases to exists in this set because $\gamma(k_1,k_2)>1$. To understand the characterisation of type B equilibrium, it is useful to give the intuition of this result by presenting the Edgeworth cycle in a market for differentiated products. To this end, we use the Figure 3 below.

Starting with $p_2 = 1$, firm 1 uses the fact that k_2 is not very large to enjoy the market share $1 - k_2$ at the security price $\rho(k_2)$ rather than fighting against $p_2 = 1$ with its "Hotelling" best reply H(1) = 1. If firm 1 sets $p_1 = \rho(k_2)$, there is no competition and the best reaction of firm 2 is to increase its price to $\delta(k_2)$, the maximum price compatible with sales of k_2. Now, both prices are at their peak and the only way to increase profit is to capture new market shares by undercutting one's opponent price.

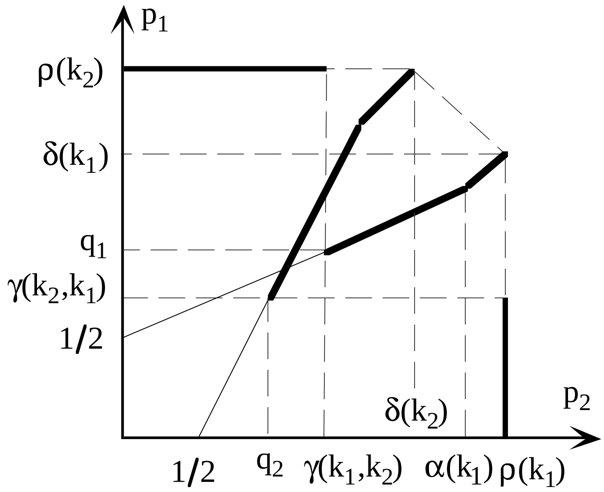

Figure 3

The next best move of firm 1 is $p_1 = H(p_2)$ (above q_1), followed by a low value $p_2 = H(p_1)$ (slightly above q_2); at this moment we are back to the beginning of the story: it is better for firm 1 to retreat over its protected share $1 - k_2$ with a high price.

According to the Nash definition in this context, the equilibrium sees firm 2 playing the pure strategy $\gamma(k_1,k_2)$ while firm 1 mixes between $\rho(k_2)$ and the lower price q_1 as described on Figure 3 above. This kind of equilibrium was identified first by Krishna [89]. Note that the symmetric vector of strategies is not an equilibrium. Indeed to make firm 2 indifferent between $\rho(k_1)$ and q_2, firm 1 would have to play the pure strategy $\gamma(k_2,k_1)$ which is strictly less than 1. This contradicts the fact that equilibrium prices are larger than 1 as established in lemma 1. As k_1 is "large", the default option appears to be never relevant for firm 2 because it involves an almost zero residual demand and thus zero profits.

The analysis of area $A_1 \backslash A_2$ is entirely symmetric. In the complement of $A_1 \cup A_2$, both $\gamma(k_1,k_2)$ and $\gamma(k_2,k_1)$ are greater than unity so that both type B equilibrium can coexist.

C) Equilibria involving completely mixed strategies

All A and B type equilibria previously mentioned exist in areas that shrinks as S gets larger. When these equilibria do not exist, completely mixed strategy equilibria must exist. The piecewise linearity of the demand functions implies that firms do not use densities in equilibrium (cf. theorem 1). In lemma A.2 of the appendix we show that when S increases, the number of atoms necessary to build an equilibrium increases and is the same for each firm.

In order to characterise a n-atom equilibrium we proceed as follows. When a firm uses **n** atoms, it has to solve **n** conditions of local optimality and **n** profit equalities using the **n** prices of his own support and the **m** probabilities over his opponent's prices. If $m < n$ then this problem has generically no solution while if m > n it has an infinity. It is therefore natural to obtain m = n in order to be able to derive each player's best replies in prices $\left(p_i^k\right)_{k=1}^n$ as a function of the vector of prices played by the other $\left(p_j^k\right)_{k=1}^n$. The operator thus obtained may then have a fixed point which will be an equilibrium candidate. An **n** atom equilibrium is a quadruple $\left(p_1^m, p_2^m, \mu_1^m, \mu_2^m\right)^{m \leq n}$ where μ_i^m is the weight put by firm **i** on her m^{th} atom p_i^m (prices are ranked by increasing order). To derive an **n** atom equilibrium of the pricing game $G(k_1,k_2)$, we consider a grid of capacity couples over the [0,1]x[0,1] area and solve numerically[8] a system of 2n – 2 polynomial equations in 2n – 2 variables. Then we check two conditions on the vector of prices derived from the system in relation to k_1, k_2 and S i.e., we eliminate some capacity points whose associated candidate equilibria violate one of those conditions.

The symmetry enables us to limit ourselves to the case where $k_1 > k_2$. The first necessary condition (cf. lemma A.2 in the appendix) states that $2k_1 - 1 > p_1^m - p_2^m > 2k_2 - 1$ for every atom **m** ; it disqualifies capacity points (k_1,k_2) exhibiting a too large differential. The reduced form of the condition reads $k_2 > g^n(k_1)$ where each g^n is an increasing and convex function. As **n** increases, more inequalities have to be satisfied, more capacity points are eliminated and the area where atomic equilibrium exist shrinks ; hence those functions satisfy $g^2 < g^3 < g^4 < g^5$ ...

The second necessary condition is related to the upper bound on prices ; it links the upper prices of the distributions to the reservation price by $p_1^n + p_2^n < 2S - 1$. Since the equilibrium prices do not critically depend on the capacity differential but on the total capacity, we study this condition on the diagonal. For a given symmetric capacity choice (k,k), we compute the symmetric candidate equilibrium $\left(p^m(k)\right)^{m \leq n}$ and the minimal reservation price for which the condition is satisfied i.e., $S_{min}^n(k) \equiv \frac{2p^n(k)+1}{2}$. The inverse of this function gives us the maximal capacity $K_{max}^n(S)$ such that points (k_1,k_2) with $k_1+k_2 \leq 2.K_{max}^n(S)$ have an **n** atoms price equilibrium at the given S. Those functions will be useful in the subsequent section.

As one could expect, the larger the capacities, the larger the prices in a candidate equilibrium. In fact, our computations show that the upper prices of a candidate equilibrium tend to infinity as capacities tend to (1,1). Now, as Proposition 2 showed that prices are

[8] It is indeed necessary as for a 5 atoms equilibrium, a system of 8 equations involving polynomials of degree 7 with more than 1500 monomials has to be solved with the Mathematica™ software.

bounded by $\rho(k_j) \equiv S-1+k_j$, capacity choices around (1,1) have no atomic price equilibria ; for that reason our numeric computations can safely stop at $k_1 = 0.99$. ♦

It is important at this step to note the reason why type B equilibrium may fail exist. The pure strategy $\gamma(k_1,k_2)$ is optimal for firm 2 only if the point $(\gamma(k_1,k_2),q_1)$ of Figure 3 above lies strictly in the "band" because otherwise the demand addressed to firm 2 at $\gamma(k_1,k_2)$ is k_2 when facing $\rho(k_2)$ and $1-k_1$ when facing q_1. This means that firm 2 has a strict incentive to raise its price until $\delta(k_2)$. So our semi-mixed candidate equilibrium is not a valid candidate. In fact, when $\gamma(k_1,k_2) > \alpha(k_1)$ (which happens when capacities are similar) there exists a completely mixed equilibrium (called type B') where both firms play two prices and where firm 1 who has the largest capacity plays the security price $\rho(k_2)$. This equilibrium shares with the semi-mixed one a very nice property: that of yielding a payoff which is independent of capacity. Indeed, in both cases firm 1 plays the security price with a strictly positive probability. Evaluating the payoff at this atom yields $\Pi_1^s \equiv (1-k_2)(S-1+k_2)$. This equilibrium is formally derived in lemma A.4 of the appendix.

Note that the description of an Edgeworth cycle in part B) of the preceding argument should not be criticised for its dynamic presentation of the static concept of Nash equilibrium. Beyond showing why there is no equilibrium in pure strategies, it helps to understand the nature of the new equilibrium. In this equilibrium where firm 1 is playing the pure strategy $\gamma(k_2,k_1)$, if firm 2 perceives a slightly larger price, it replies aggressively for sure while if it perceives a slightly lower price, it plays the security price for sure. We follow here the purification argument of Harsanyi [73]. According to this interpretation our mixed strategy equilibrium satisfies the no-regret property for the firm and therefore escape the standard criticism of this equilibrium concept. Moreover, the experimental study of Brown-Kruse & al. [94] suggest that disequilibrium adjustment process (called Edgeworth cycle in their paper) or mixed strategy equilibria are the most robust theoretical explanation of the observed pricing pattern in a Bertrand-Edgeworth oligopoly game.

Building on our characterisation of price equilibria we may state Corollary 1 which is instrumental for the resolution of the capacity game. By deriving an explicit formula for the second period profit of one firm, we will be able to construct our focal SPE. It is an equivalent to Proposition 1 in KS which states that the high capacity firm is paid according to its Stackelberg follower payoff i.e., as a function of the small capacity firm.

COROLLARY 1

When pure strategy equilibria do not exist, there always exists a mixed strategy equilibrium in which firm i, the large capacity firm, is paid $\Pi_i = (1-k_j)(S-1+k_j)$.

Proof Assume firm 2 is the large capacity firm and consider first firm 2's profit in a type B equilibrium. It is computed from either price in the support of its equilibrium strategy. At the atom $\rho(k_1) \equiv S - 1 + k_1$, firm 2's demand is $1 - k_1$ so that $\Pi_2 = (1 - k_1)(S - 1 + k_1)$. In pricing games where type B equilibria do not exist, we can build a completely mixed strategy equilibrium where firm 2 plays its security strategy with a positive probability. Lemma A.4 provides an explicit derivation of the two atom equilibrium. Using Lemma A.3 and the procedure described in Lemma A.4, equilibrium involving more than two atoms and one firm naming its security price can be build as well . ♦

4) Capacity commitment and Cournot outcomes

Going backward is difficult in the game G because $G(k_1,k_2)$ has often several price equilibria as shown in proposition 3. The focal SPE of our model involves symmetric choices by the firms. The corresponding equilibrium outcomes replicate those of a monopoly owning the two firms: the market is shared evenly, there is no excess of capacity, global surplus and firms' joint profits are maximised, and consumer surplus is minimised.

PROPOSITION 4

Committing to the Hotelling equilibrium quantities 1/2 and naming the monopoly price S – 1/2 is a Subgame Perfect Equilibrium.

Proof A downward deviation cannot be profitable since we assumed that the monopoly profit of a constrained firm $(S - \varepsilon - k)k$ is increasing up to $\frac{S-\varepsilon}{2} > \frac{1}{2}$. To deter an upward deviation, we define the continuation price equilibrium of G(k,1/2) to be of type B or B' (cf. proposition 3). It yields a profit of $(S - \frac{1}{2})\frac{1}{2} - \varepsilon k$ for the deviant. This is not profitable because of the supplementary capacity cost. ♦

Theorem 2 confirms the intuition according to which prices are in a sense "too low" in the standard Hotelling equilibrium, i.e. there is room for relaxing competition. Capacity precommitment allows firms to sell exactly their Hotelling demands, but at a much higher price. At this equilibrium, firms are on their local monopoly profit curve so that, contrary to the standard equilibrium result, prices directly depend on S. Notice that this price is the highest one that ensures full market coverage since it leaves the marginal consumer located in 1/2 with zero surplus. Since a monopoly owner of both firms would implement precisely this outcome, the issue of this non-cooperative competition seems to be collusive. However, this subgame perfect equilibrium is not unique as the following proposition shows.

PROPOSITION 5

When the capacity cost is negligible, two kinds of SPE coexist
i) Complementary capacities equilibria (CCE) *where the total capacity exactly covers the market and each firm enjoys a minimum share; furthermore, the price equilibrium is in pure strategies.*
ii) Overlapping capacities equilibria (OCE) *where the total capacity exceed the market size (overlapping capacities), the difference in the capacity choices is limited and the price equilibrium is in mixed strategies.*

Proof ***i)*** **Complementary capacities equilibria**

Without loss of generality, consider capacities choices $(m,1-m)$ with $m \geq 1/2$. The profit function of firm 1 and 2 on the intervals $[0;m]$ and $[0;1-m]$ respectively is $(S - \varepsilon - k)k$. We have shown when proving proposition 1 that this function is increasing up to $\frac{S-\varepsilon}{2}$, which is therefore an upper limit to **m** in order to deter downward deviations (for a large S, this limit is not binding).

An upward deviation $k_1 > m$ by firm 1 can only lead to a price equilibrium of type B or C. There is no possibility for a type A equilibrium because it requires both capacities to be large and since $m \geq 1/2$, the capacity choice of firm 2 is smaller than the required limit $\Phi(S,1/2)$ (as derived in Lemma A.1 in appendix). To deter an upward deviation of firm 1, we define the continuation price equilibrium of $G(k_1,1-m)$ to be of type B or B' which yields a net profit for firm 1 of $(S-m)m - \varepsilon k_1$; this is a non profitable deviation because of the supplementary cost of capacity.[9]

If $m < \Phi(S,1/2)$, an upward deviation $k_2 > 1-m$ by firm 2 can only be followed by a type B or C equilibrium and we apply the same trick as for firm 1 to deter this upward deviation. Whenever $m \geq \Phi(S,1/2)$ firm 2's payoff is almost nil, hence it has an incentive to deviate to the large capacity $\Phi(S,1/2)$ in order to play the Hotelling price equilibrium and earn a net profit of $1/2 - \varepsilon\Phi(S,1/2)$. Whenever ε is less than half (the relevant condition for the original Hotelling model), the solution of the equation $1/2 - \varepsilon\Phi(S,1/2) = [S - \varepsilon - 1 + m](1-m)$ give us a bound on **m** which is obviously less than one. We may conclude that any pair (k_1,k_2) such that $k_1+k_2=1$ can be sustained as part of a SPE as long as each firm obtains at least its "Hotelling profit".

ii) **Overlapping Capacities Equilibria**

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[9] As mixed strategies enable large prices, type C equilibria provide too large payoffs and cannot be used to sustain our candidate SPE.

Consider now a candidate SPE outcome (m_1,m_2) such that $m_1+m_2>1$. To prove that it is not a SPE, we must consider deviation to (m_1,k_2) or (k_1,m_2) and look at the worst price equilibrium for the deviant ; if the deviation is still profitable then (m_1,m_2) cannot be sustained as a SPE.

Claim If (m_1,m_2) is such that no type C equilibria exists in $G(m_1,m_2)$, then this choice is not part of a SPE.

If the price equilibrium in $G(m_1,m_2)$ is of type A, firms earn a profit independent of their capacity choices. Therefore, each has an incentive to reduce capacity since the cost ε is positive (almost nil is exactly what is needed). If the price equilibrium is of type B or B', the payoff of one firm, say **i**, in the pricing game is $\Pi^d(m_j)$; by choosing $k_i = 1 - m_j$, firm **i** sets itself in a non overlapping situation and achieves $(S - 1 + m_j)(1 - m_j) = \Pi^d(m_j)$ with a lower cost of capacity installation, thus it will deviate.

This artefact is our main instrument to rule out "unwanted" equilibria ; we also obtain a first result: overlapping capacities are sustainable as SPE choices only if capacities are not too dissimilar. The point (m_1,m_2) must satisfy $m_1 > g^n(m_2)$, i.e. there exists type C price equilibria.

We now build a SPE with capacity choices (m_1,m_2) such that the equilibrium of $G(m_1,m_2)$ is an **n** atom one. This couple must satisfy $m_2 > g^n(m_1)$ (by symmetry of G, we can always assume $m_1 < m_2$) and $m_1 + m_2 < 2\,K^n_{max}(S)$; those conditions taken together define an upper bound on capacities. We now define the strategies out of the equilibrium path: at (k_i,m_j), we define pricing strategies to be the pure strategy $\gamma(k_i,m_j)$ for firm **j** while firm **i** mixes between $\rho(m_j)$ and the lower price $\frac{1+\gamma(k_i,m_j)}{2}$ (type B equilibrium). Firm **i** obtains $\Pi^d(m_j)$ and to deter the deviation k_i, it must be less than $\Pi^n(m_1,m_2)$, the profit accruing to firm **i** at the **n** atom equilibrium.

This latter function mostly depends on the total capacity, therefore we may study the previous condition on the diagonal. We thus solve $\Pi^n(k,k) - \varepsilon k \geq [S - (1-k) - \varepsilon](1-k)$ in the variable S to get $S \leq S^n_{max}(k,\varepsilon) \equiv \dfrac{\Pi^n(k,k) - \varepsilon k}{1-k} + 1 - k + \varepsilon$. The numerical computation is performed for ε = 0 as we are studying the case of almost nil capacity cost. Then, we can invert $S^n_{max}(k,0)$ to obtain a lower bound $K^n_{min}(S)$ on capacities which is compatible with the upper bound $K^n_{max}(S)$ derived in proposition 3 (the above simplification is therefore valid up to small numerical errors).

Contrarily to type **i)** SPE, the capacity combinations that appear as SPE of type **ii)** are functions of S. Figure 4 below summarises our result: the various $K^n_{min}(S)$ and $K^n_{max}(S)$ functions are plotted for n = 2, 3, 4 and 5. Consider for example S = 4. There exists a symmetrical[10] SPE with a 2 atoms price equilibrium if the capacity is between .81 and .85 and SPE with a 3 atoms price equilibrium if the common capacity is between .6 and .61. For S = 6, there exists SPE with 2, 3, 4 or 5 atoms price continuation equilibria. The larger S, the larger the number of mixed strategy equilibria in the pricing games and therefore the larger the number of possible capacity overlapping SPE equilibria. ♦

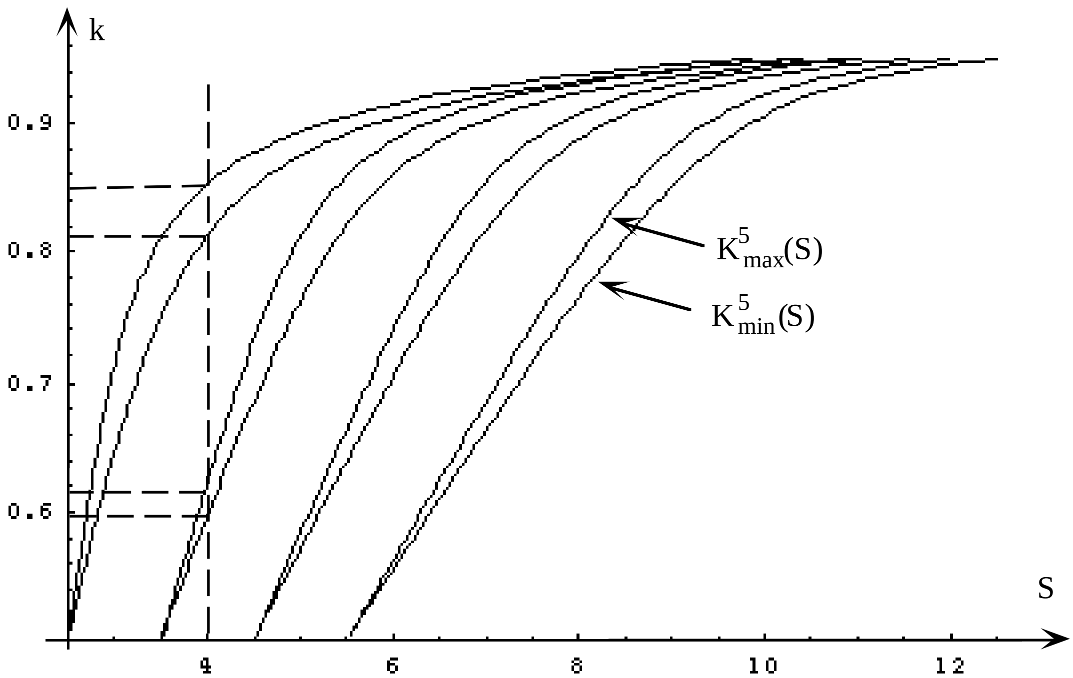

Figure 4

Obviously, the existence of our overlapping capacity SPE is eased by the fact the capacity cost is negligible. The following theorem studies the robustness of our equilibria to the level of the capacity cost.

PROPOSITION 6

If the capacity cost is larger than 1/4 only CCE subgame perfect equilibria exist.

Proof We have shown that a type **ii)** SPE exists for the symmetric capacity **k** only if $S^n_{min}(k) \le S \le S^n_{max}(k,\varepsilon)$. As the latter function is decreasing in ε, $S^n_{min}(k) = S^n_{max}(k,\varepsilon)$ has a solution $\varepsilon^n(k)$ and for any $\varepsilon > \varepsilon^n(k)$, the candidate SPE is removed. The equation to solve is

$$S^n_{min}(k) = \frac{\Pi^n(k,k) - \varepsilon k}{1-k} + 1 - k + \varepsilon \Rightarrow \varepsilon^n(k) \equiv \frac{\Pi^n(k,k) - \left[S^n_{min}(k) - (1-k)\right](1-k)}{2k-1}.$$

10 Whenever a symmetrical n-atom equilibrium exists, there also exists asymmetrical ones for all capacity choices with the same mean and satisfying $m_i > g^n(m_j)$.

The ε^n functions satisfy $\varepsilon^2 > \varepsilon^3 > \varepsilon^4 > \varepsilon^5$ and are concave, decreasing, reaching zero at $\frac{1}{4}$. It is clear that for every $\varepsilon > 1/4$, no type **ii)** equilibria remains. ♦

The following comments are in order. First, note that in the SPE's involving exact market coverage, both firms are on their monopolist's profit curve. This perfectly illustrates why firms may benefit from capacity precommitment. Indeed, the main feature of the Hotelling model is that firms may enjoy local monopolies around their locations. However, in the absence of capacity constraints, they cannot prevent price competition to take place because their monopolist's natural markets overlap. Although positive mark-ups are preserved in equilibrium, price competition is damaging to the firms. This is clearly seen by observing that in the Hotelling equilibrium, prices do not depend on S. In other words, firms fail to capture a large part of the consumers' surplus. The main virtue of capacity precommitment is precisely to avoid this failure. Indeed, through capacity precommitment, firms are now able to capture the greatest part of the consumers' surplus. In particular, their equilibrium payoffs depend positively on S.

Second, the existence of equilibria involving excess capacities is mainly due to the existence of multiple equilibria in the price subgame where firms fight for market shares. However, it remains true that in these equilibria, prices are always above the Hotelling prices and are positively related to S. The corresponding payoffs are also positively linked to the reservation price. Thus, the qualitative conclusion remains: through capacity commitment, firms systematically sustain higher prices.

Summing up, *whatever the subgame perfect equilibrium considered, we are led to conclude that capacity precommitment enables firms to take full advantage of the local monopoly structure which is inherent to the Hotelling model.*

In the case of an homogenous product, KS show that firms tend to avoid destructive price competition through capacity commitment. In Proposition 5 and 6, we have extended their result to the case of horizontal differentiation. The nicest feature of the KS result is that it provides a theoretical foundation for Cournot competition that allows for an explicit price mechanism. We now show that a similar result obtains in our model of horizontal differentiation.

PROPOSITION 7

The equilibrium quantities of a Cournot game in the Hotelling model correspond to the Complementary Capacities SPE of the capacity precommitment game.

Proof In the Cournot game, firms supply quantities q_1 and q_2 to an otherwise competitive market: i.e. prices must clear the market (see Grilo & Mertens [99] for a foundation). If the proposed quantities q_1 and q_2 do not cover the market, there is excess demand and the prices increase until supply equals demand on each side of the market i.e., $q_i = S - p_i$ for **i** = 1,2. This situation is unstable for S> 2, since at least one firm has an incentive to increase its quantity above the complement of the other. If now the proposed quantities q_1 and q_2 exceed the market size, there is excess aggregate supply and at least one of the price, say p_1, must be nil on this competitive market. Therefore firm 1 has a profitable deviation by offering a quantity slightly less than $1 - q_2$ to be on its monopoly profit curve. The candidates for a Cournot equilibrium are (q, 1 − q) with q ≤ 1/2. As a firm can supply q − ε to guarantee the price S − q + ε, the market-clearing prices have to be S − q and S − 1 + q in a SPE. Without loss of generality, we assume that firm 1 offers **q**, thus sells less than 1/2 in equilibrium. Hence, p_1 cannot be nil because it would attract at least one half of the consumers, thereby implying an excess demand. Firm 2 cannot profitably deviate to a larger quantity than 1 − q because it would face a zero price (one price is nil and by the preceding argument, it must be its price). Firm 1 may profitably deviate to some Q larger than q but still less than 1/2. Since there is excess supply, p_2 is nil, thus firm 1 sells all of Q and the consumer located at x = Q must be indifferent in equilibrium so that $p_1 = 1 - 2Q$. The profit Q(1 − 2Q − ε) reaches a maximum of $\frac{(1-\varepsilon)^2}{8}$ at $\frac{1-\varepsilon}{4}$ to be compared with q(S − q − ε). Since q ≤ 1/2, the only relevant root is $q^* \equiv \frac{2S-2\varepsilon-\sqrt{2(S-\varepsilon)^2-(1-\varepsilon)^2}}{4} > 0$.

Thus, the Cournot equilibria must feature exact market coverage (q, 1 − q) with q larger than this lower bound[11] q^*.♦

5) Comments and Conclusion

In this paper, we made a first step toward reconciling the two lines of research initiated by the Bertrand paradox i.e., models of capacity commitment and product differentiation. We have shown that the nature of equilibria in capacity-constrained pricing games with product differentiation significantly differ from the case of homogeneous products. Multiple equilibria prevail and firms never use densities on the support of a mixed strategy equilibrium. Using the Hotelling model, we have shown that product differentiation does not preclude the use of capacity commitment as a device to relax price competition. On the contrary, firms always limit production capacities in equilibrium. The main interest of such strategies is to sustain

[11] This lower bound is different from that derived in theorem 3 but both are small so that our equivalence applies for the most likely sharing of the market.

higher prices in order to appropriate a larger part of the consumer surplus. The mechanism at work replicates that of the market for an homogeneous product. However, the specificity of differentiated markets is to allow for many equilibria in the pricing game. Therefore, beside Cournot-like outcomes, there exist equilibria exhibiting excess capacities and completely mixed pricing equilibria.

These result has been established in a particular framework that calls for discussion. For instance, it is well known that the nature of the rationing rule plays a central role in pricing models with capacity constraints. Davidson & Deneckere [86] show that the KS result entirely rests upon their assumption of efficient rationing. In the present analysis also, the particular rule of rationing is instrumental in achieving clear-cut results. Any alternative to the efficient rule would result in a lower residual demand addressed to the "high" price firm. However, the local monopoly structure of the model does not crucially depend on the rationing rule. Therefore, it is intuitive that equilibria would have the same qualitative features, it would only take a capacity cost larger than 1/4 to eliminate Overlapping capacities SPE. Moreover, in our setting, the efficient rationing rule may be viewed as a rather natural one when interpreting the Hotelling model as a spatial model. In this case indeed, the rule basically organises rationing on a "first arrived-first served" basis.

We consider a market in which the location of firms are exogeneously fixed at the extremities of the market. As mentioned in the introduction this assumption is motivated by its implications for price competition: as product differentiation is maximised, this is the case where the firms have the lowest incentives to further relax price competition. Having proven the strength of the incentives to use capacity precommitment, we can expect that the same forces would apply when firms are located inside the market. If firms are not located inside the first and third quartiles (see next point), then each has a protected market so that the incentives to play the pricing game as a local monopolist are reinforced. All demands functions characterised in sub-section 3.1 have an added elastic monopolist demand term of the form $\min\{d_1,S-p_1\}$ where d_1 is the location of firm 1 inside the market.

More generally though, the robustness of our result to alternative location patterns is not easy to trace. Indeed, even without capacity constraints, pure strategy equilibrium may fail to exist in the linear Hotelling model when firms are located inside the first and third quartiles (see Osborne & Pitchick [87] for a characterisation of mixed strategies equilibria). No doubt this possibility severely complicates the characterisation of equilibria in pricing subgames. Two remarks are in order in this respect.

First, it should be noted that the presence of capacity constraints may help to restore the Hotelling equilibrium for locations inside the first and third quartiles (see Wauthy [96]). At the same time, inside locations will tend to make upward deviations less profitable, because they could imply less favourable residual demands. We may therefore suspect that with inside locations, there is less scope for excess capacity choices whereas exact market coverage remains most attractive.

Second, under quadratic transportation costs, there always exists a pure strategy equilibrium in pricing games without capacity constraints. In a related paper (Boccard and Wauthy [2000]), we consider Hotelling pricing games with quadratic transportation costs where only one firm is capacity constrained. We study location patterns in this particular case and are able to show that the classical maximal differentiation does not hold in the presence of the capacity constraint. Instead firms tend to move to the centre of the market, because the constrained firm wishes to maximises the surplus (i.e. minimise transportation costs) over the restricted market area delimited by its capacity. This is exactly the strategy that would follow a monopolist.

Last, the present paper combines two means by which firms may relax price competition: exogenous product differentiation and endogenous capacity precommitment. A natural step forward is to endogenize the first device by dealing simultaneously with product differentiation issues as well as some form of capacity constraints. The question will then become: should firms use both devices simultaneously or prefer one against the other ? The answer will obviouslydepend on the specificity of industries. However, our conjecture is that if product differentiation increases aggregate welfare, firms will tend to differentiate while relying on capacity constraints to appropriate the largest possible part of that surplus. For instance in the Hotelling model, we conjecture that firms located at the quartiles of the line segment, each with a production capacity of one half would be sustained as a SPE of a location-capacity-price game. Although being quite specific, Boccard and Wauthy [2000] points in this direction. In contrast, Boccard and Wauthy [99] shows for the case of vertical differentiation that the presence of a capacity constraint drastically weakens incentives to differentiate by quality. In this case indeed, there is nothing to gain in terms of total surplus by choosing to differentiate: i.e. by choosing a lower quality.

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Appendix

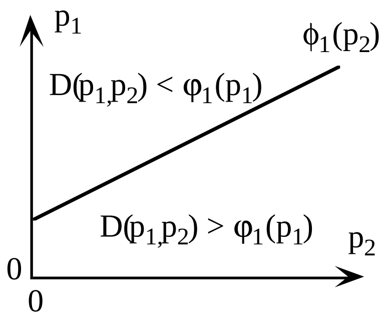
PROOF OF THEOREM 1

Consider any price subgame $\Gamma(C_1,C_2)$. Under Bertrand competition, there exists a unique pure strategy SPE. Under Bertrand-Edgeworth competition, an SPE exists, the support of a mixed strategy price SPE is finite and prices are larger than those of the Bertrand equilibrium.

Proof We shall show that payoffs' are continuous which then guarantees the existence of an equilibrium, possibly in mixed strategies. Let F_1, F_2 be the cumulative functions of the mixed strategy used by firm 1 and 2 in a Nash equilibrium following the choices of C_1 and C_2. The well known result according to which an increasing convex function is unbounded can be applied to the decreasing concave function D to show that there exists a price $\overline{p}$ beyond which demand is nil. Hence the supports of the strategies F_1 and F_2 are included in $[0;\overline{p}]$. For i = 1, 2 let f_i be the derivative of F_i if it exists while $\left(p_i^k\right)_{k\leq N_i}$ denotes the set of discontinuity points of F_i and $\alpha_i^k \equiv F_i(p_i^k) - \lim_{x\to p_i^{k-}} F_i(x)$ are their associated mass. Lastly $\overline{p}_i$ and $\underline{p}_i$ denote the supremum and infimum of the support of F_i.

Step 1 Delimitation of the support of equilibrium strategies

If $D(0,0) > \varphi_1(0)$ as on Figure A1 below (large $\dot{C}_1$) then the equation $\varphi_1(p_1) = D(p_1,p_2)$ has a unique solution $p_1 = \phi_1(p_2) > 0$ since $\frac{\partial D(p_1,p_2)}{\partial p_1} < 0$ and φ_1 is non decreasing. Now $\frac{\partial D(p_1,p_2)}{\partial p_2} > 0$ implies that ϕ_1 is increasing thus $D\big(\phi_1(0),p_2\big) > \varphi_1\big(\phi_1(0)\big)$ for all p_2. For any distribution F_2, the profit to firm 1 on $\left[0;\phi_1(0)\right]$ is the increasing function $\Pi_1^C(.)$ which means that in a Nash equilibrium of $\Gamma(C_1,C_2)$ firm 1 puts no mass below $\phi_1(0)$. The same holds true for firm 2 with $\phi_2(0)$ by symmetry. To reach the same conclusion in the analysis of pure Bertrand competition simply observe that Π_i^B is increasing over the domain $D(p_i,p_j) > \varphi_i(p_i)$; thus in a Bertrand equilibrium no firm puts mass below $\phi_i(0)$ for i = 1,2.



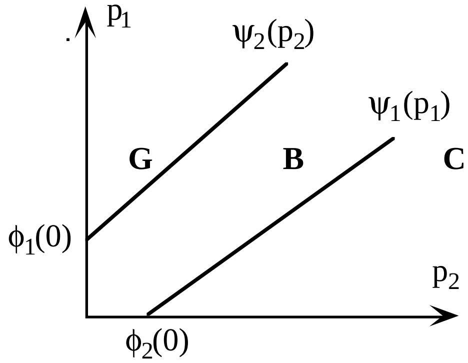


Figure A1

Figure A2

By translating the origin we can consider w.l.o.g. to the case depicted on Figure A2 where $D(0,0) \leq \varphi_1(0)$. Let $p_2 = \psi_1(p_1)$ be the solution to $\varphi_1(p_1) = D(p_1, p_2)$. We derive $\dot{\varphi}_1 = \frac{\partial D}{\partial p_1} + \frac{\partial D}{\partial p_2}\dot{\psi}_1$ thus $\dot{\psi}_1 = \frac{\dot{\varphi}_1 - \frac{\partial D}{\partial p_1}}{\frac{\partial D}{\partial p_2}} \geq \frac{-\frac{\partial D}{\partial p_1}}{\frac{\partial D}{\partial p_2}} \geq 1$. By symmetry we can assume w.l.o.g. that $\psi_2(.)$ is a positive and increasing function with slope larger than 1. Hence, as shown on Figure A2, $\psi_1(.)$ and $\psi_2(.)$ never cross but they may be parallel (as will be the case in the Hotelling model with capacity commitment analysed in sections 3 to 7). Area **G, B** and **C** are delimited by the graphs of $\psi_1(.)$ and $\psi_2(.)$ as shown on Figure A2.

Whenever $p_1 > \psi_2(p_2)$ (area **G** on Figure A2), firm 2 rations consumers letting firm 1 recover a fraction $\lambda(p_1, p_2)$ of the residual demand $D(p_2, p_1) - \varphi_2(p_2)$. The demand addressed to firm 1 is thus $G(p_1, p_2) \equiv D(p_1, p_2) + \lambda(p_1, p_2)\big(D(p_2, p_1) - \varphi_2(p_2)\big)$. Using **A4)** $\frac{\partial \lambda}{\partial p_1} \leq 0$ (decreasing spillover effect), we obtain $\frac{\partial G}{\partial p_1} = \frac{\partial D}{\partial p_1} + \lambda(p_1, p_2)\frac{\partial D}{\partial p_2} + \frac{\partial \lambda}{\partial p_1}\big(D(p_2, p_1) - \varphi_2(p_2)\big) \leq \frac{\partial D}{\partial p_1} + \frac{\partial D}{\partial p_2} \leq 0$. Since spillovers areas do not overlap we also have $D(p_1, p_2) < G(p_1, p_2) < \varphi_1(p_1)$. Hence $\Pi_i^G(p_i, p_j) \equiv p_i G(p_i, p_j) - C_i\big(G(p_i, p_j)\big)$ is the profit function over area **G** and satisfies $\frac{\partial \Pi_1^G}{\partial p_1} > \frac{\partial \Pi_1^B}{\partial p_1}$.

Step 2 Properties of the various payoff functions

Π_i^B is increasing over area **C** as $p_i < \dot{C}_i\big(D(p_i, p_j)\big)$. Over areas **B** and **G** where $p_i \geq \dot{C}_i\big(D(p_i, p_j)\big)$ assumption **A 3)**: $0 \leq \frac{\partial^2 D}{\partial p_i \partial p_j} p_i + \frac{\partial D}{\partial p_j}$ guarantees that $\frac{\partial^2 \Pi_i^B}{\partial p_i \partial p_j} = \frac{\partial D}{\partial p_j}\Big(1 - \frac{\partial D}{\partial p_i}\ddot{C}_i(D)\Big) + \frac{\partial^2 D}{\partial p_i \partial p_j}\big(p_i - \dot{C}_i(D)\big) > 0$ since taking $\dot{C}_i$ constant is the worst case. We need concavity of Π_i^B and Π_i^G. To prove the first we only need to apply condition **A3)**: $2\frac{\partial D}{\partial p_i} + \frac{\partial^2 D}{\partial p_i^2} p_i \leq 0$ to $\frac{\partial^2 \Pi_i^B}{\partial p_i^2} = \frac{\partial D}{\partial p_i}\Big(2 - \frac{\partial D}{\partial p_i}\ddot{C}_i(D)\Big) + \frac{\partial^2 D}{\partial p_i^2}\big(p_i - \dot{C}_i(D)\big)$ because taking $\dot{C}_i$ constant is the worst case. To prove that Π_i^G is concave we study $M = 2\frac{\partial G}{\partial p_i} + \frac{\partial^2 G}{\partial p_i^2} p_i =$

$$2\left(\frac{\partial D}{\partial p_i} + \lambda(p_i,p_j)\frac{\partial D}{\partial p_j} + \frac{\partial \lambda}{\partial p_i}\left(D(p_j,p_i) - \varphi_j(p_j)\right)\right) + p_i\left(\frac{\partial^2 D}{\partial p_i^2} + \frac{\partial \lambda}{\partial p_i}\frac{\partial D}{\partial p_j} + \lambda\frac{\partial^2 D}{\partial p_i^2} + \frac{\partial^2 \lambda}{\partial p_i^2}\left(D(p_j,p_i) - \varphi_j(p_j)\right) + \frac{\partial \lambda}{\partial p_i}\right.$$

Using **A3)** : $\frac{\partial^2 D}{\partial p_i \partial p_j} p_i + \frac{\partial D}{\partial p_j} \leq -2\frac{\partial D}{\partial p_i} - \frac{\partial^2 D}{\partial p_i^2} p_i$ yields

$M \leq \left(2\frac{\partial D}{\partial p_i} + p_i \frac{\partial^2 D}{\partial p_i^2}\right)(1-\lambda) + \left(2\frac{\partial \lambda}{\partial p_i} + p_i \frac{\partial^2 \lambda}{\partial p_i^2}\right)\left(D(p_j,p_i) - \varphi_j(p_j)\right) + \frac{\partial \lambda}{\partial p_i} p_i \left(\frac{\partial D}{\partial p_j} + \frac{\partial D}{\partial p_j}\right) < 0$ since **A3)** also yields $2\frac{\partial D}{\partial p_i} + \frac{\partial^2 D}{\partial p_i^2} p_i \leq 0$ and **A4)** gives $2\frac{\partial \lambda}{\partial p_i} + p_i \frac{\partial^2 \lambda}{\partial p_i^2} \leq 0$.

By construction firms sales vary continuously from areas **G** to **B** and from **B** to **C**. The integral with respect to an increasing distribution function is also continuous thus payoffs are continuous which guarantees existence of an equilibrium by a standard fixed point theorem. Notice also that whenever cost is continuously differentiable then $\dot{\Pi}_i^C(p_i) = \varphi_i(p_i) = \frac{\partial \Pi_i^B(p_i,\psi_i(p_i))}{\partial p_i}$ as $p_i = \dot{C}_i\left(D(p_i,\psi_i(p_i))\right)$ i.e., payoff is continuously differentiable at the frontier between areas **B** an d **C**. Yet payoffs are not continuously differentiable as soon as firms use atoms since $\frac{\partial \Pi_i(p_i,F_j)}{\partial p_i}$ jumps upward at $p_i = \psi_i\left(p_j^k\right)$ when atom p_j^k passes from area **B** to area **G**. As a consequence $p_i = \psi_i\left(p_j^k\right)$ cannot be part of a Nash equilibrium F_i if F_j has an atom at p_j^k.

<u>Step 3</u> The Bertrand equilibrium is unique and is a lower bound to the price distributions in a Bertrand-Edgeworth equilibrium.

Let $p_i = r_i^B(p_j)$ be the unique solution of $0 = \frac{\partial \Pi_i^B(p_i,p_j)}{\partial p_i} = D(p_i,p_j) + \left(p_i - \dot{C}_i\left(D(p_i,p_j)\right)\right)\frac{\partial D(p_i,p_j)}{\partial p_i}$. It is indeed unique as Π_i^B is concave. As we showed that $\frac{\partial^2 \Pi_i^B}{\partial p_i \partial p_j} > 0$ over the domain $p_i \geq \dot{C}_i\left(D(p_i,p_j)\right)$, r_i^B is increasing. Using **A3)**: $\frac{\partial^2 D}{\partial p_i \partial p_j} p_i + \frac{\partial D}{\partial p_j} \leq -2\frac{\partial D}{\partial p_i} - \frac{\partial^2 D}{\partial p_i^2} p_i$ we obtain $\dot{r}_i^B \leq 1$ (taking $\dot{C}_i$ constant is the worst case). Hence the solution $\left(p_i^B, p_j^B\right)$ of $\left\{p_i^B = r_i^B(p_j^B), p_j^B = r_j^B(p_i^B)\right\}$ is the unique Bertrand equilibrium in pure strategies.

We now prove that this pair is the unique equilibrium of the price game. Observe on Figure A2 that over $[0;\phi_1(0)]$, $\Pi_1(p_1,F_2)$ is the average of Π_1^B satisfying $\frac{\partial^2 \Pi_1^B}{\partial p_1 \partial p_2} > 0$ and of the increasing function Π_1^C over **C** and that $\Pi_1^B(p_1,\psi_1(p_1)) = \Pi_1^C(p_1)$ (demand continuity) thus

$$\frac{\partial \Pi_1(p_1,F_2)}{\partial p_1} = \int_0^{\psi_1(p_1)} \frac{\partial \Pi_1^B}{\partial p_1} dF_2(p_2) + \left(1 - F_2\left(\psi_1(p_1)\right)\right) \frac{\partial \Pi_1^C(p_1)}{\partial p_1} + f_2\left(\psi_1(p_1)\right)\dot{\psi}_1(p_1)\left(\Pi_1^B(p_1,\psi_1(p_1)) - \Pi_1^C(p_1)\right)$$

$$\geq \int_0^{\psi_1(p_1)} \frac{\partial \Pi_1^B(p_1,p_2)}{\partial p_1} dF_2(p_2) \geq \frac{\partial \Pi_1^B(p_1,0)}{\partial p_1}$$

Hence $\Pi_1(p_1,F_2)$ is increasing up to $\beta_1^1 \equiv \min\left\{\phi_1(0); r_1^B(0)\right\}$ meaning that F_1 puts no mass below β_1^1. The same result holds in the Bertrand case since Π_1^B is increasing over area **C**. By symmetry for firm 2, F_2 puts no mass below $\beta_2^1 \equiv \min\left\{\phi_2(0); r_2^B(0)\right\}$ (and likewise in the Bertrand equilibrium). Observe now on Figure A2 that over $\left[\beta_1^1; \psi_2(\beta_2^1)\right]$, $\Pi_1(p_1,F_2)$ is the average of Π_1^B over **B** and the increasing Π_1^C over **C** thus $\Pi_1(p_1,F_2)$ is increasing up to $\beta_1^2 \equiv \min\left\{\phi_2(\beta_2^1); r_1^B(\beta_2^1)\right\}$. This process of symmetrical adjustments leads to $\underline{p}_i \geq p_i^B$ and $\underline{p}_j \geq p_j^B$.

In the Bertrand case we can start from above using $\frac{\partial \Pi_i^B(\bar{p},F_j)}{\partial p_i} = 0 + \left(\bar{p} - \dot{C}_i(0)\right)\frac{\partial D}{\partial p_i} < 0$ and prove in a similar fashion that $\bar{p}_i \leq r_i^B(\bar{p}) < \bar{p}$. In the Bertrand-Edgeworth competition Π_1^G applies over area **G** and since $\frac{\partial \Pi_1^G}{\partial p_1} > \frac{\partial \Pi_1^B}{\partial p_1}$ we cannot say that $\frac{\partial \Pi_1(p_1,F_2)}{\partial p_1} \leq \frac{\partial \Pi_1^B(p_1,\bar{p})}{\partial p_1}$ to reach the same conclusion. It is only for Bertrand competition that we reach the Bertrand pair $\left(p_i^B, p_j^B\right)$ from above to conclude that the equilibrium must be in pure strategies.

<u>Step 4</u> Equilibrium distributions have a finite support in the Bertrand-Edgeworth competition

$$\Pi_1(p_1,F_2) = \int_0^{\psi_2^{-1}(p_1)} \Pi_1^G(p_1,p_2)dF_2(p_2) + \int_{\psi_2^{-1}(p_1)}^{\psi_1(p_1)} \Pi_1^B(p_1,p_2)dF_2(p_2) + \Pi_1^C(p_1)\left(1 - F_2(\psi_1(p_1))\right)$$

$$\frac{\partial \Pi_1(p_1,F_2)}{\partial p_1} = \int_0^{\psi_2^{-1}(p_1)} \frac{\partial \Pi_1^G(p_1,p_2)}{\partial p_1} dF_2(p_2) + \int_{\psi_2^{-1}(p_1)}^{\psi_1(p_1)} \frac{\partial \Pi_1^B(p_1,p_2)}{\partial p_1} dF_2(p_2) + \dot{\Pi}_1^C(p_1)\left(1 - F_2(\psi_1(p_1))\right)$$

because demands are continuous from one area to the other.

$$\frac{\partial^2 \Pi_1(p_1,F_2)}{\partial p_1^2} = \int_0^{\psi_2^{-1}(p_1)} \frac{\partial^2 \Pi_1^G(p_1,p_2)}{\partial p_1^2} dF_2(p_2) + \int_{\psi_2^{-1}(p_1)}^{\psi_1(p_1)} \frac{\partial^2 \Pi_1^B(p_1,p_2)}{\partial p_1^2} dF_2(p_2) + \ddot{\Pi}_1^C(p_1)\left(1 - F_2(\psi_1(p_1))\right)$$

$$+ \left(\frac{\partial \Pi_1^G}{\partial p_1} - \frac{\partial \Pi_1^B}{\partial p_1}\right) f_2\left(\psi_2^{-1}(p_1)\right)\dot{\psi}_2^{-1}(p_1) + \left(\frac{\partial \Pi_1^B}{\partial p_1} - \dot{\Pi}_1^C(p_1)\right) f_2(\psi_1(p_1))\dot{\psi}_1(p_1)$$

where the last two terms are respectively weakly positive (spillover effect at $p_2 = \psi_2^{-1}(p_1)$) and weakly negative (exactly zero if cost is strictly convex i.e., when $\varphi_1 = \dot{C}_1^{-1}$).

In order that firm 1 plays according to a density around a price p_1, $\frac{\partial \Pi_1(p_1,F_2)}{\partial p_1} = 0$ must hold locally thus $\frac{\partial^2 \Pi_1(p_1,F_2)}{\partial p_1^2} = 0$ must likewise hold locally. We have already shown that Π_1^B and Π_1^G were concave, thus $f_2\left(\psi_2^{-1}(p_1)\right) > 0$ must hold whenever $\ddot{\Pi}_1^C(p_1) = 0$. This will happen if $\dot{C}_1$ is a step function with finitely many jumps because the point p_1 can always be chosen between the jumps. As a consequence firm 2 must use a density around $\psi_2^{-1}(p_1)$. Starting from the upper bound p_1^* of the density played by firm 1 we deduce that firm 2 plays a density at $\psi_2^{-1}(p_1^*)$. By symmetry (if $\dot{C}_2$ is also a step function) firm 1 must be playing a density around $\psi_1^{-1}\left(\psi_2^{-1}(p_1^*)\right)$. As seen on figure A2 above this process leads to prices below the Bertrand ones, a contradiction.

If firm 1 uses an infinite number of atoms in equilibrium there must exist a price p_1 such that a neighbourhood of p_1 contains infinitely many atoms thus $\frac{\partial^2 \Pi_1(p_1,F_2)}{\partial p_1^2} = 0$ must hold for otherwise the continuous payoff function could have infinitely many maximum in that neighbourhood. This implies that firm 2 must be playing infinitely many atoms around $\psi_2^{-1}(p_1)$. The previous reasoning therefore carry on to this case. The support of a Nash equilibrium distribution is thus finite when marginal cost functions are stepwise. What is at the heart of this proof is the existence of the Bertrand area characteristic of differentiated markets. In the homogeneous case studied by Kreps & Scheinkman, it is reduced to a line and our iterative procedure cannot be applied to rule out densities.

Any marginal cost function $\dot{C}_1$ can be approached by a step function to yield a payoff function $\hat{\Pi}_1^\varepsilon$ whose distance from the original Π_1 in the supremum norm over $[0;\bar{p}]\times[0;\bar{p}]$ is less than ε. Let $\left(F_1^*,F_2^*\right)$ be a Nash equilibrium of the original game and let $\left(p_1^k\right)_{k=1}^{n_1}$ be the argmax of $\hat{\Pi}_1^\varepsilon\left(p_1,F_2^*\right)$. For every ε there exists α such that $\left|p_1 - p_1^k\right| > \alpha$ implies $\hat{\Pi}_1^\varepsilon(p_1,F_2^\varepsilon) < \hat{\Pi}_1^\varepsilon(p_1^k,F_2^*) - \varepsilon$ (this comes from the fact that the points p_1^k are isolated) thus $\Pi_1(p_1,F_2^\varepsilon) < \Pi_1(p_1^k,F_2^*)$. As a consequence the support of F_1 is included in $\left[p_1^B;\bar{p}\right] \setminus \bigcup_{k=1}^{n_1}\left[p_1^k \pm \alpha\right]$. The same argument applies for firm 2. We proved in step 1 that there are no indices k,l such that $p_1^k = \psi_2(p_2^l)$ or $p_2^l = \psi_1(p_1^k)$ thus we may take ε small enough to guarantee that rectangles $\left[p_1^k \pm \alpha\right]\times\left[p_2^l \pm \alpha\right]$ never intersect the graphs of ψ_1 and ψ_2 i.e., they belong to the interior of area **B, G** or **C**. Over each $\left[p_1^k \pm \alpha\right]$, $\frac{\partial \Pi_1(p_1,F_2^*)}{\partial p_1}$ is a finite sum of integrals of either $\frac{\partial \Pi_1^G}{\partial p_1}$, $\frac{\partial \Pi_1^B}{\partial p_1}$ or $\dot{\Pi}_1^C$ over intervals of the form $\left[p_2^l \pm \alpha\right]$, thus it is continuously differentiable and has finitely many zeroes leading to finitely many best replies to F_2^*: the support of a Nash equilibrium is finite. ♦

LEMMA A.1

Derivation of the cut-off between aggressive pricing (Lemma 1) and monopoly behaviour and existence of the A type equilibria (proposition 3)

Proof **i)** The profit of the Hotelling best reply $\varphi_1(p_2) = \begin{cases} \frac{1+p_2}{2} & \text{if } p_2 < \alpha(k_1) \\ p_2 + 1 - 2k_1 & \text{if } p_2 \geq \alpha(k_1) \end{cases}$, the associated profit is $\begin{cases} \underline{\Pi}_1(p_2) \equiv \dfrac{(1+p_2)^2}{8} & \text{if } p_2 < \alpha(k_1) \\ \overline{\Pi}_1(p_2) \equiv k_1[p_2 + 1 - 2k_1] & \text{if } p_2 \geq \alpha(k_1) \end{cases}$ while that associated with the security strategy $\rho(k_2)$ is $\Pi^d(k_2) \equiv [S-1+k_2](1-k_2)$. The solution to $\Pi^d(k_2) = \underline{\Pi}_1(p_2)$ is denoted $x \equiv \sqrt{8[S-1+k_2](1-k_2)} - 1$ while that of $\Pi^d(k'_2) = \overline{\Pi}_1(p_2)$ is denoted $y \equiv \frac{[S-1+k'_2](1-k'_2)}{k_1} - 1 + 2k_1$. Observe that if S is too small then **x** is negative and the security strategy is never used; also the bound **x** is useful for large values of k_2 as Π^d is decreasing; solving in k_2 the inequality $x \leq \alpha(k_1) \Leftrightarrow \sqrt{8[S-1+k_2](1-k_2)} - 1 \leq 4k_1 - 1$ leads to $k_2 \geq \Lambda(S, k_1) \equiv \frac{2-S+\sqrt{S^2-8k_1^2}}{2}$ having eliminated the negative solution. Letting $\gamma(k_1, k_2) \equiv \begin{cases} y & \text{if } k_2 < \Lambda(S, k_1) \\ \max\{0, x\} & \text{otherwise} \end{cases}$, we obtain the best reply function of firm 1 as $BR(p_2) \equiv \begin{cases} \rho(k_2) & \text{if } p_2 \leq \gamma(k_1, k_2) \\ \varphi_1(p_2) & \text{if } p_2 > \gamma(k_1, k_2) \end{cases}$. The best reply of firm 1 to a low price p_2 is the default option $\rho(k_2)$ and above the threshold **x**, the optimal play becomes H(.). For the smaller capacity k'_2, the cut-off value is **y** and the optimal play above **y** is $a(k_1,.)$.

ii) We characterise the area of the capacity space $[0;1]^2 \cap \{k_2 + k_1 > 1\}$ where the pure strategy equilibrium (type A in proposition 4) exists. The condition $\gamma(k_1,k_2) < 1$ is equivalent to $\{k_2 < \Lambda(S, k_1) \text{ and } y < 1\}$ or $\{k_2 \geq \Lambda(S, k_1) \text{ and } x < 1\}$. When $\gamma(k_1,k_2) = y$, we solve $1 > 2k_1 - 1 + \frac{[S-1+k_2](1-k_2)}{k_1}$. Using the only meaningful root, we derive: $k_2 > \Phi(S, k_1) \equiv \frac{2-S+\sqrt{S^2-8k_1(1-k_1)}}{2}$. Note that $\Phi(S,k_1) < k_2 < \Lambda(S,k_1)$ make sense only for $k_1 < 1/2$. When $\gamma(k_1,k_2) = x$, we solve $\sqrt{8[S-1+k_2](1-k_2)} - 1 < 1$ and we get $k_2 > \Phi(S,1/2) = \frac{2-S+\sqrt{S^2-2}}{2} \in [1/\sqrt{2};1]$ valid for S larger than 2. Area A_1 is thus defined by its lower contour $\Phi(S,k_1)$ over $[0;\frac{1}{2}]$ and the constant $\Phi(S,1/2)$ over $[\frac{1}{2};1]$. Area A_2 is symmetrically defined and since $\Phi(S,1/2) > 1/2$, area A where the Hotelling equilibrium exists is the square $[\Phi(S,1/2);1] \times [\Phi(S,1/2);1]$. Yet, we cannot claim that it is the only equilibrium of area A because profit functions are not concave. ♦

LEMMA A.2

In a completely mixed strategy equilibrium of G(k_1,k_2), firms use the same number of atoms **n** *and* $\forall\, k \le n-1,\ 1-2k_2 < \underline{p}_1^k - \underline{p}_2^k < 2k_1 - 1$.

Proof We use the table of points formed by the distributions $\left(p_1^k\right)_{k=1}^m$ and $\left(p_2^k\right)_{k=1}^n$ (cf. Figure A3 below) ; we speak of lines when p_1 is fixed, of columns when p_2 is fixed and of the "diagonal" for the pairs $\left(p_1^k, p_2^k\right)_{k\ge 1}$. The equilibrium being in completely mixed strategies assume w.l.o.g. $m \ge n > 1$.

Claim 1 The pair $\left(\underline{p}_1, \underline{p}_2\right)$ of minimal atoms lies in the band (cf. point α on Figure A3)

If $\underline{p}_1 < \underline{p}_2 + 1 - 2k_2$, then the whole line $\left(\underline{p}_1, p_2^k\right)_{k=1}^m$ lies under the band and $\Pi_1(F_2,.) = \Pi_1^C(F_2,.)$ which is locally increasing meaning that $\underline{p}_1$ cannot be part of an equilibrium. We have the symmetric result for firm 2. In the Hotelling model, the same reasoning implies that every line and every column has at least one point in the band because $\Pi_1^C(F_2,.) = k_1 p_1$ and $\Pi_1^G(F_2,.) = (1-k_2)p_1$

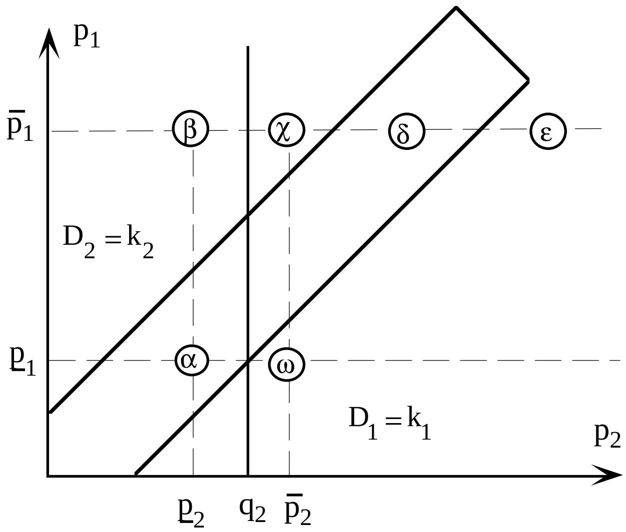


Figure A3

Claim 2 ω and β lie strictly outside the band

Let $\bar{p}_i$ be the second atom of F_i for i = 1, 2 and $q_2 \equiv \underline{p}_1 - 1 + 2k_2$. Since F_1 has a countable support, $\Pi_2(F_1,.)$ cannot be constant over an interval thus not over $\left[\underline{p}_2, \bar{p}_2\right]$, therefore it must strictly decrease after $\underline{p}_2$ and then strictly increase up to $\bar{p}_2$. Yet since $\Pi_2(F_1,.)$ is concave at $\underline{p}_2^-$ (we do not exclude $\underline{p}_2 = q_2$ yet), the only way to make it increasing again is to move into the positive externality area when reaching $\bar{p}_2$. Consequently ω = $\left(\underline{p}_1, \bar{p}_2\right)$ (and may be some other points in the ω–χ column) must lie strictly in the lower triangle i.e., $\bar{p}_2 > q_2$. The case for β is symmetrical.

Claim 3 α is strictly interior to the band

If $\underline{p}_2 = q_2$, $D_2(F_1,.)$ is locally increasing at $\underline{p}_2^+$ because β lies strictly in the upper triangle, a contradiction. The case for the upper frontier of the band is symmetrical.

Claim 4 All points $\left(p_1^k, p_2^k\right)_{k=1}^{n-1}$ are strictly interior to the band

The second "diagonal" point $\left(\overline{p}_1, \overline{p}_2\right)$ could be either χ, δ or ε on Figure A3 above. If it were χ, then the ω–χ column would have no point in the band and $\Pi_2(F_1,.)$ would be locally increasing at $\overline{p}_2$. Likewise, if it were ε, the β–ε line would have no point in the band. Thus $\left(\overline{p}_1, \overline{p}_2\right)$ is δ (it could lie on one frontier). By the argument of claim 2, if **n** >2 then $\left(\overline{p}_1, p_2^3\right)$ must lie strictly in the lower triangle ; likewise $\left(p_1^3, \overline{p}_2\right)$ must lie strictly in the upper triangle. Now if δ were at the left edge of the band, then $\Pi_1(F_2,.)$ would be locally increasing at $\overline{p}_1^+$ and if δ were at the right edge of the band $\Pi_2(F_1,.)$ would be locally increasing at $\overline{p}_2^+$. The reasoning applies for all diagonal points whenever there is one more atom above. Only the last diagonal point $\left(p_1^n, p_2^n\right)$ could lie on one frontier of the band.

Claim 5 The distributions have the same number of atoms.

Suppose m = n + 1. Consider first the case where $\left(p_1^n, p_2^n\right)$ is not interior to the band. Since m > n there is another point on the right with $p_2^{n+1} \leq \rho(k_1)$. If $\left(p_1^n, p_2^n\right)$ were on the right frontier then $\Pi_2(F_1,.)$ would be strictly increasing over $\left[p_2^n; p_2^{n+1}\right]$ a contradiction. Thus $\left(p_1^n, p_2^n\right)$ is on the left frontier as on Figure A4 below. Still, if $p_1^n < \rho(k_2)$, $\Pi_1(F_2,.)$ is strictly increasing over $\left[p_1^n; \rho(k_2)\right]$, a contradiction, therefore $p_1^n = \rho(k_2)$ as on Figure A4. To prove that m = n, observe that $D_2(F_1,.)$ has the same definition over the whole interval $\left]r_2; q_2\right]$ and being concave on that interval it cannot have two strict maximisers. The case for $\left(p_1^n, p_2^n\right)$ interior to the band is depicted on Figure A5. It is similar because the change in $D_2(F_1,.)$ from r_2 to q_2 is the switch from the slowly decreasing duopolistic demand to the more decreasing monopoly demand thus reinforcing the impossibility to have another argmax of $\Pi_2(F_1,.)$ beyond r_2. The case for m > n + 1 is now easy because $\Pi_2(F_1,.)$ must have the same shape over $\left[p_2^n; p_2^{n+1}\right]$ or over $\left[p_2^{n+1}; p_2^{n+2}\right]$ which leads to a contradiction. ♦

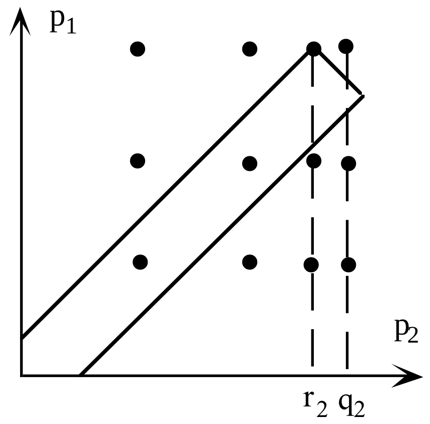


Figure A4

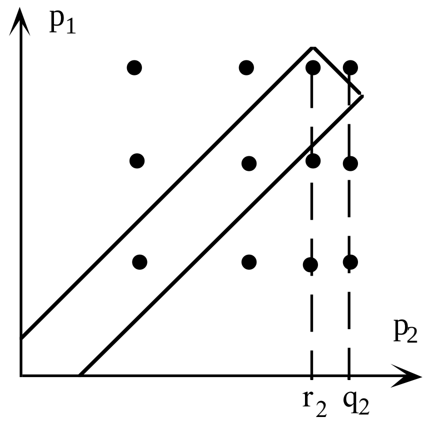


Figure A5

LEMMA A.3

The nature of the equilibria with ***n*** *atoms in the pricing game $G(k_1,k_2)$.*

Proof Consider the distributions $\left(p_1^m,\mu_1^m\right)_{m\le n}$ and $\left(p_2^m,\mu_2^m\right)_{m\le n}$ of an equilibrium with **n** atoms on each side. By lemma 3 we can write for $m \le n$:

$$\Pi_1(F_2,p_1^m) = p_1^m\left[(1-k_2)\sum_{h<m}\mu_2^h + \mu_2^m\frac{1-p_1^m+p_2^m}{2} + k_1\sum_{h>m}\mu_2^h\right] \qquad \text{(E3)}$$

We consider first an equilibrium where all diagonal points are interior to the band i.e., $p_2^n < \rho(k_1)$ and $p_1^n < \rho(k_2)$. In that case, the first order condition (FOC) of local optimality applies with equality for all $j \le n$ by lemma 3.

$$\frac{\partial\, \Pi_1(F_2,p_1^m)}{\partial\, p_1^m} = 0 \Leftrightarrow \quad 2(1-k_2)\sum_{h<m}\mu_2^h + \mu_2^m(1-2p_1^m+p_2^m) + 2k_1\sum_{h>m}\mu_2^h = 0 \qquad \text{(E4)}$$

By virtue of (E4), the profit at the optimum is $\Pi_1(F_2,p_1^m) = \dfrac{\mu_2^m(p_1^m)^2}{2}$. The same first order condition for firm **2** at its mth atom leads to the following system with the constants M_1 and M_2 adequately defined:

$$\begin{cases} \mu_2^m(1-2p_1^m+p_2^m) = -2k_1\sum_{h>m}\mu_2^h - 2(1-k_2)\sum_{h<m}\mu_2^h = -M_1 \\ \mu_1^m(1-2p_2^m+p_1^m) = -2k_2\sum_{h>m}\mu_1^h - 2(1-k_1)\sum_{h<m}\mu_1^h = -M_2 \end{cases}$$

$$\Rightarrow \qquad 3\mu_1^m\mu_2^m p_1^m = 3\mu_1^m\mu_2^m + 2\mu_1^m M_1 + 2\mu_2^m M_2$$

$$\Rightarrow \qquad p_1^m = 1 + \frac{4}{3\mu_2^m}\left[(1-k_2)\sum_{h<m}\mu_2^h + k_1\sum_{h>m}\mu_2^h\right] + \frac{2}{3\mu_1^m}\left[(1-k_1)\sum_{h<m}\mu_1^h + k_2\sum_{h>m}\mu_1^h\right]$$

(E5)

We also have the **n** equations for the prices charged by firm 2. Having eliminated the prices, the number of unknowns is reduced from 4n to 2n. Since $\mu_i^n = 1 - \sum_{m<n}\mu_i^m$ for i = 1,2, we can use the vectors $u \equiv \begin{pmatrix}\mu_1^m \\ \mu_2^h\end{pmatrix}_{\substack{m<n \\ h<n}}$ of 2n − 2 unknowns and the 2n − 2 equations system is obtained by equating profit for each firm at each of the atoms it plays in equilibrium i.e. 0 = $X(u) \equiv \begin{pmatrix}\mu_2^n(p_1^n)^2 - \mu_2^m(p_1^m)^2 \\ \mu_1^n(p_2^n)^2 - \mu_1^h(p_2^h)^2\end{pmatrix}_{\substack{m<n \\ h<n}}$.

This system can be reduced to a polynomial one with as many equations as unknowns, it has a finite number of solutions (cf. Theorem 1). Except for the 2-atoms case (see Lemma A.4), we have not been able to check unicity beyond variation of the starting point of our algorithm. The algorithm we are proposing is only an equilibrium selection, not the equilibrium correspondence. It must be noted that all equations are fractional and can thus be reduced to polynomial equations with a maximum exponent of 7 (independently of **n**). Furthermore, if we count k_1 and k_2 as variables, each equation contains 60 monomials for n = 2, 247 monomials for n = 3, 686 monomials for n = 4 and 1533 monomials for n = 5. It must be noted that even for n = 2, the *Mathematica* software is not able to solve this polynomial system. We have therefore programmed an algorithm for this purpose.

Since p_1^n is proportional to $\frac{1}{\mu_2^n}$, the profit $\mu_2^n(p_1^n)^2$ decreases with μ_2^n, thus by choosing μ_1^n and μ_2^n nearby 1 i.e. **u** near 0, we obtain X(u) << 0. The Taylor expansion of the differentiable function X is $X(u + du) = X(u) + dX.du$ where dX is the Jacobian of X evaluated at **u**. We approach a solution by following the path of optimal growth i.e., we choose $\mathbf{du} = -\delta.(dX)^{-1}.X(u)$ where δ is chosen to enable a rapid but certain convergence of the numerical computation. If $p_2^n = \rho(k_2)$ then $p_1^n = \delta(k_2)$ and vice versa. One implication is that (E4) must be satisfied only as an inequality at $\rho(k_2)^-$ and $\delta(k_2)^-$ thus there is more slack for the choice of the probabilities. This permits a candidate equilibrium to pass more easily the supplementary constraint that the diagonal points of the equilibrium distribution must lie in the band. We show this in the next lemma for the 2 atoms case.

The width of the band is defined as K = $2(k_1+k_2-1)$. By proposition 3, firm 1's support is included in $\left[\frac{(1-k_2)}{k_1}(S-1+k_2);S-1+k_2\right]$ thus the range of prices has a length of

$\frac{(S-1+k_2)K}{2k_1}$. The reasoning of lemma 4 has shown that the distance from p_1^1 to p_1^3 is at least K, thus if there are 2n+1 atoms, the necessary range is at least nK. We obtain a bound on the number of atoms with S: $\frac{(S-1+k_2)K}{2k_1} \geq nK \Rightarrow 2n+1 \leq \frac{S+k_2+k_1-1}{k_1}$. ♦

LEMMA A.4

Analysis of the two atoms price equilibrium

Proof We first analyse the "interior" equilibrium as a particular case of the previous lemma and then we derive the existence of the two atoms price equilibrium where the firm with the largest capacity plays her security price.

For a two atoms equilibrium with prices $\left(\underline{p}_1, \overline{p}_1\right)$ and distribution $(\mu_1, 1-\mu_1)$, we let $\beta_i \equiv \frac{1-\mu_i}{\mu_i}$ for i = 1, 2 so that system (E5) for the lower and upper atoms reads:

$$3\underline{p}_1 = 3 + \frac{4k_1}{\mu_2}(1-\mu_2) + \frac{2k_2(1-\mu_1)}{\mu_1} = 3 + 4\beta_2 k_1 + 2\beta_1 k_2 \qquad \text{(E6)}$$

$$\text{and} \qquad 3\overline{p}_1 = 3 + \frac{4(1-k_2)\mu_2}{(1-\mu_2)} + \frac{2(1-k_1)\mu_1}{(1-\mu_1)} = 3 + 4\frac{1-k_2}{\beta_2} + 2\frac{1-k_1}{\beta_1} \qquad \text{(E7)}$$

The equality of the profits $\Pi_1(F_2, \underline{p}_1) = \frac{\mu_2(\underline{p}_1)^2}{2}$ and $\Pi_1(F_2, \overline{p}_1) = \frac{(1-\mu_2)(\overline{p}_1)^2}{2}$ simplifies to $\mu_2(3 + 4\beta_2 k_1 + 2\beta_1 k_2)^2 = (1-\mu_2)\left(3 + 4\frac{1-k_2}{\beta_2} + 2\frac{1-k_1}{\beta_1}\right)^2$

$$\Leftrightarrow \qquad 3 + 4\beta_2 k_1 + 2\beta_1 k_2 = \sqrt{\beta_2}\left(3 + 4\frac{1-k_2}{\beta_2} + 2\frac{1-k_1}{\beta_1}\right) \qquad \text{(E8)}$$

We derive β_1 as the largest (i.e., the meaningful) root $f(\beta_2,k_1,k_2)$ of the second degree equation (E8). By symmetry for firm **2**, we get $\beta_2 = f(\beta_1,k_2,k_1)$. It is now clear that an equilibrium of the pricing game is a fixed point of $f\left(f(.,k_1,k_2),k_2,k_1\right)$. Since $\underline{p}_1 < \overline{p}_1$ and profits are proportional to $\mu_1(\underline{p}_1)^2$ and $(1-\mu_1)(\overline{p}_1)^2$, it must be true that $\mu_1 > 1/2$, thus $\beta_2 < 1$ and symmetrically $\beta_1 < 1$. Those supplementary conditions are helpful to analyse the large capacity case. Observe that, independently of the capacities, if β_2 tends to 0, the second degree equation tends to $2k_2[\beta_1]^2 - 4\frac{1-k_2}{\sqrt{\beta_2}}[\beta_1] = 0$. Its positive solution diverges and by symmetry, we obtain $f(\beta_1,k_2,k_1) \xrightarrow[\beta_1 \to 0]{} +\infty$. Now, since β_2 is bounded, the constant in

(E8) tends to zero when k_1 approaches unity, thus $\beta_1 = f(\beta_2, k_1, k_2)$ $= \frac{-B + \sqrt{B^2 - 4AC}}{2A} \xrightarrow[k_1 \to 1]{} 0$. Plugging this in the preceding result, we see that β_2 = $f(\beta_1, k_2, k_1)$ diverges, a contradiction with the constraint $\beta_2 < 1$.

We can therefore undertake numerical computations without worrying about the behaviour at the corner (1,1). As **f** is analytically known, we have been able to compute the roots of $f\left(f(., k_1, k_2), k_2, k_1\right)$ for a lattice of capacities such that $k_i + k_j > 1$; it appears that this function is always decreasing, thus there is at most one equilibrium. The conditions provided by lemma 4 arising from (E6) and (E7) are

$$3(1-2k_2) < 3 + 4\beta_2 k_2 + 2\beta_1 k_1 - \left(3 + 4\beta_1 k_1 + 2\beta_2 k_2\right) < 3(2k_1 - 1)$$

$$\Leftrightarrow \quad \frac{3}{2}(1-2k_2) < \beta_2 k_2 - \beta_1 k_1 < \frac{3}{2}(2k_1 - 1) \qquad \text{(E9)}$$

$$\text{and } \frac{3}{2}(1-2k_2) < \frac{1-k_1}{\beta_2} - \frac{1-k_2}{\beta_1} < \frac{3}{2}(2k_1 - 1) \qquad \text{(E10)}$$

They allow us to eliminate couples with high capacity differential. Our computations show that the upper prices $\bar{p}_i$ and $\bar{p}_j$ increase with capacities; thus $\bar{p}_i \le \rho(k_j) = S - 1 + k_j$ is violated for large capacities i.e., the point $\left(\bar{p}_i, \bar{p}_j\right)$ leaves the "band" as described in lemma 5. Consequently, atomic equilibrium will never exist for capacities nearby unity.

We now derive the constrained equilibrium (type B') where the high capacity firm 1 uses its security strategy $\rho(k_2)$ in equilibrium. For this two atoms equilibrium with bottom prices q_1 and q_2 and distribution $(\mu_1, 1-\mu_1)$, we let $\beta_i \equiv \frac{1-\mu_i}{\mu_i}$ for i = 1, 2 so that system (E5) for the lower atom reads:

$$q_1 = 1 + \frac{4k_1}{3\mu_2}(1-\mu_2) + \frac{2k_2(1-\mu_1)}{3\mu_1} = 1 + \frac{4}{3}\beta_2 k_1 + \frac{2}{3}\beta_1 k_2 \qquad \text{(E11)}$$

A symmetric equation holds for firm 2 ; the interiority condition for q_1 and q_2 is still (E9) which amount to require sufficiently low values for β_1 and β_2. We consider potential deviation at the large prices $\rho(k_2)$ and $\delta(k_2)$ for firm 1 and 2 respectively. At $\rho(k_2)$, an upward deviation of firm 1 leads to a monopolistic demand and thus $\rho(k_2)$ is optimal. For a downward deviation, $\left.\frac{\partial\, \Pi_1(F_2, p_1)}{\partial\, p_1}\right|_{\rho(k_2)^-} > 0 \Leftrightarrow \mu_2(1-k_2) + (1-\mu_2)\frac{1-2\rho(k_2)+\delta(k_2)}{2} > 0 \quad \Leftrightarrow$

$$\beta_2 < \frac{2(1-k_2)}{S - 3(1-k_2)} \qquad \text{(E12)}$$

As $q_1 < \rho(k_2) - K$, the local condition at $\delta(k_2)^-$ for firm 2 is trivial because the demand is always constant. At $\delta(k_2)^+$, $\Pi_2(F_1, p_2) = p_2\left[\mu_1(1-k_1) + (1-\mu_1)(S-p_2)\right]$. The FOC is $\left.\frac{\partial \Pi_2(F_1,p_2)}{\partial p_2}\right|_{\delta(k_2)^+} < 0 \Leftrightarrow \mu_1(1-k_1) + (1-\mu_1)(S - 2\delta(k_2)) < 0 \quad \Leftrightarrow \quad \beta_1 < \frac{1-k_1}{S-2k_2}$

(E13)

These constraints are compatible with (E10). The equality of payoffs at q_1 and $\rho(k_2)$ for firm 1 leads to $\frac{\mu_2(q_1)^2}{2} = \Pi_1^S \Leftrightarrow \frac{\mu_2(3+4\beta_2 k_1 + 2\beta_1 k_2)^2}{18} = (S-1+k_2)(1-k_2)$

$$\Leftrightarrow \quad (3+4\beta_2 k_1 + 2\beta_1 k_2)^2 = 18(S-1+k_2)(1-k_2)(1+\beta_2) \qquad \text{(E14)}$$

while for firm 2, we obtain $\frac{\mu_1(q_2)^2}{2} = \delta(k_2)\left(\mu_1(1-k_1) + (1-\mu_1)k_2\right)$

$$\Leftrightarrow \quad (3+4\beta_1 k_2 + 2\beta_2 k_1)^2 = 18(S-k_2)\left(1-k_1+\beta_1 k_2\right) \qquad \text{(E15)}$$

Using the numerical method previously introduced, we are enable to solve system (E14-E15) in an unconstrained manner and then eliminate those solution which violate conditions (E12-E13). For small values of S, we obtain equilibria for the capacity pairs where the type B equilibrium does not exist. Yet when S increases, the numerical solution to (E14-E15) does not pass the test anymore meaning that a price equilibrium is a minimum of 3 atoms that enable a fair share of weights among them. This is consistent with the formula on the maximum number of atoms derived at the end of lemma A.3. ♦