One-to-One Matching Problems with Location Restrictions

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Abstract

This paper introduces a novel set of one-to-one matching problems: matchings subject to location restrictions. When scarcity of matching locations exists some agents may want to form a new partnership without being able to implement it. In this general setting we develop two stability concepts, direct and (coalition) exchange* stability, akin to Gale Shapley stability and exchange stability (Alcalde, 1995) respectively. We show that coalition-exchange* stability is a refinement of direct stability. When no location scarcity exists then direct stability is equivalent to Gale Shapley stability and coalition-exchange* stability is equivalent to requiring both exchange stability (Alcalde, 1995) and Gale Shapley stability. We show that the set of coalition-exchange* stable matchings is a superset of the farsighted core, and equal to the farsighted core if locations are not scarce and the matching problem is individually rational. The paper also shows that an exchange* stable set can not be a strict subset of a farsighted stable set and provides an example of a roommate problem in which no farsighted stable set exists while an exchange* stable set does exist. Finally, the paper obtains that deciding whether the farsighted core of an individually rational roommate problem exists is NP-complete.

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1 Introduction

Consider a roommate problem such that two students prefer to share a room rather than sticking to their current roommates. On their own, these students can only do so if there is a room available to them. Indeed, when an assignement gives a student the right to occupy a given room, and all rooms are occupied by others, these students cannot, on their own, enforce a new matching in which they occupy a room together. More generally, the size of the set of available rooms (matching locations) can effectively restrict the possible matchings that a set of agents can enforce over the current assignment. When this is the case, it is natural to consider the set of available matching locations as a primitive of the matching problem, alongside the agents and their preferences. A stability concept that is defined on such a matching problem should then take these primitives as given.

The first contribution of this paper is to formally introduce a, possibly scarce, set of matching locations into one-to-one matching problems. Doing so, we generalize one-to-one matching problems by possibly requiring that a match between agents must happen at a matching location. We introduce a finite set of matching locations and the notion of a location mapping as a function that assigns agents to a matching location, if any, providing them with the right to be matched at that location. We then require that a match between two agents happens if and only if both are assigned to the same location by a location mapping.

The second contribution of our paper is to study stability in this setting by analyzing the incentives of (a group of) agents to change the current 'status quo' matching. In order to do so, we first analyze how a set of agents can alter the composition of any given matching. This is operationalized through the idea of enforceability. We introduce two different concepts of enforceability. First, we say that a set of agents can **directly enforce** one matching over an initial assignment if they can reassigning the matching *locations* amongst them that are exclusively under their control. Second, we say that a set of agents can **exchange* enforce** one matching over an initial assignment if they can reassign themselves the *rights* that are exclusively under their control. The key difference between the two enforceability conditions is that direct enforceability requires that the agents must control all rights assigned to a location, while exchange* enforceability only requires that they exchange location rights, thereby potentially 'forcing' other agents to accept a different partner of the (deviating) set. In other words, exchange* enforceability

implies that if a partner x of an agent y exchanges her matching right with someone else, say agent z, then agent y has no say in this exchange, even if it means she will become matched to agent z. We also allow agents to exchange their current location right for any available (unassigned) locations right as it seems natural to let agents, when they are allowed to exchange matching location rights, to perform this swap. From this it logically follows that direct enforceability implies exchange* enforceability but not vice versa: exchange* enforceability allows for more possible alterations of a given matching by some set of agents than direct enforceability. The enforceability notions are then used to define dominance relationships, **direct dominance** and **exchange* dominance**, and stability concepts, **direct stability** and **coalition-exchange* stability**. A matching is directly (coalition-exchange*) stable if it is not directly (exchange*) dominated by any other matching. A matching is **exchange* stable** if it is not exchange* dominated by a pair or singleton of agents. Direct stability is closely related to **Gale Shapley stability**. Gale Shapley stability is an ex ante stability[1] concept in the sense that *before* a matching is *implemented*, no single or pair of agents (*a blocking pair or individual*) would like to deviate. Once a matching is implemented (ex-post), two agents may want to deviate but cannot if they cannot find a matching location to do so. Gale Shapley stability is thus equivalent to direct stability when assuming that the set of matching locations is never scarce. Exchange* stability is different from the notion of **exchange stability** introduced by Alcalde (1995). He defines a matching to be *exchange stable* if there does not exist an *exchange blocking pair*: no two agents can be made better off by exchanging their current matching position. The implicit assumption behind this concept is that there are no unassigned matching locations (rights) which a set of agents can exchange their current matching rights for. In other words, the set of matching locations must be limited. Our paper is rooted in the same spirit as Morrill (2010) who studies the roommate problem and asks: 'ex post, what types of coalitions will be able to block a given assignment?', recognizing that agents face two restrictions: 1. the set of rooms is limited (the amount of rooms is exactly half the amount of students) and 2. bilateral approval is needed to dissolve a match between two roommates. Morrill's (2010) set up then naturally leads to finding Pareto optimal matchings. Morrill (2010) notes that when the current assignment can be dissolved unilaterally, then (Gale Shapley) stability is a natural stability concept. Our paper argues that direct (exchange*) stability is a natural solution concept when rights cannot (can) be exchanged. Our first result (proposition 1) is that coalition-exchange* stability is a refinement of direct stability. At

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[1]This meaning of ex ante stability is to be distinguished from Kesten and Unver (2015).

first hand, this seems to be at odds with the literature in view of Alcalde's result (1995) that Gale Shapley stability and exchange stability are mutually independent stability concepts. We show that this is due to the fact that these stability concepts (implicitly) assume different location restrictions: scarcity in the case of exchange stability and no scarcity in the case of Gale Shapley stability. When one considers the set of matching locations and associated matching rights as a primitive of the matching problem then this independence result disappears. We subsequently show (proposition 2) that when matching locations are not scarce then direct stability is equivalent to Gale Shapley stability and coalition-exchange* stability is equivalent to requiring Gale Shapley stability and exchange stability simultaneously.

The third contribution of the paper is to provide an interesting link between exchange* dominance and **indirect dominance**, a concept introduced by Harsanyi (1971) and later formalized by Chwe (1994) in order to study deviations from a current 'state' when agents do not care about the immediate consequences of their actions but rather care about the final outcome after other agents have reacted to their initial reaction. The **farsighted core** of a matching problem is the set of all matchings that are not indirectly dominated by some matching. In Theorem 1 we show that whenever a matching exchange* dominates some other matching and all agents who see their match change find their new partner acceptable, then it also indirectly dominates this matching. Intuitively, if two agents wish to exchange their partners but would need the consent of the latter to do so, then they could perform this swap in two steps: in step 1 they leave their current partner and in step 2 they propose to match to complete the swap. On this path of indirect dominance, all agents will always agree to the proposed changes given the status quo matching. But, if agents are allowed to swap their current partners, these two steps can be done in one. Such indirect dominance path can not exist if the agents who were forced into a new partnership would rather be single than matched to their new partner in a swap they did not initiate. In Corollary 1 we show that the set of coalition-exchange* stable matchings of individually rational matching problems is a superset of the farsighted core and equivalent to the farsighted core if there is not location scarcity. It is well known that in this setting the farsighted core, and hence the set of (coalition) exchange* stable matchings, is often empty and if it exists, it must be a singleton. For this reason the literature (not assuming any restrictions on the set of matching locations) has introduced alternative stability concepts to study farsightedly stable matchings. A popular stability concept is the **farsighted stable**

set (FSS)[2]. A set of matchings is a FSS if every matching outside the set is indirectly dominated by some member of the set (external stability) and if the matchings of the set do not indirectly dominate each other. Equivalently we define an **exchange* stable set** (ESS) as a set of matchings that do not exchange* dominate each other and all outside matchings are exchange* dominated by some matching in the set. We then find (proposition 3) that an exchange stable set can never be a proper subset of a farsighted stable set and sometimes is a superset of a farsighted stable set. In addition we provide an example of a matching problem without a FSS but for which there exists an ESS.

The computer science literature paid ample attention to the computational complexity question of determining whether a given matching problem admits a Gale Shapley stable[3] or exchange stable matching. It is known (Manlove 2013 and Irving 2008) that deciding whether a (one-to-one) matching is both exchange stable (à la Alcalde 1995) and Gale Shapley stable is NP complete. *The last contribution of this paper* exploits the link between exchange* dominance and indirect dominance to say something about the complexity of finding farsightedly stable matchings in one-to-one matching problems when the set of location rights is not scarce, that is when coalition-exchange* stability is equivalent to exchange stability combined with Gale Shapley stability: while deciding whether a matching is Gale Shapley stable is solvable in polynomial time, deciding whether a one-to-one matching problem admits a unique farsighted matching (the farsighted core) is NP-complete (corollary 2).

The rest of the paper is organized as follows. Section 2 introduces one-to-one matching problems with location restrictions. Section 3 introduces direct stability and (coalition)-exchange* stability and discusses the relationship between these concepts. Section 4 analyzes the relationship between exchange* dominance and indirect dominance. Section 5 deals with the computation complexity question of deciding whether a given matching problem admits a farsighted stable matching. Section 6 concludes.

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[2] See Ray and Vohra (2014) for a recent discussion of this concept.

[3] Determining whether a given roommate problem with strict preferences admits a stable assignment is solvable in polynomial time (see for instance Irving, 1985 and Ronn, 1990, for a discussion).

2   One-to-one matching problems with location restrictions

2.1   The primitives

A one-to-one matching problem, or roommate problem, is a triple (L, N, P). L is a finite set of locations, where a specific location is denoted by l where $l \in L$, N is a finite set of agents and P is a preference profile specifying for each agent $i \in N$ a *strict* preference ordering over N. That is, $P = \{P(1), ..., P(i), ..., P(n)\}$, where $P(i)$ is agent i's strict preference ordering over the agents in N including herself. For instance, $P(i) = 4, 5, i, 2, ...$ indicates that agent i prefers agent 4 to agent 5 and she prefers to remain alone rather than get matched to anyone else. We denote by $\mathcal{L}$ and $\mathcal{N}$ the cardinality of L and N respectively. We denote by R the weak orders associated with P. We write $j \succ_i k$ if agent i strictly prefers j to k, $j \sim_i k$ if i is indifferent between j and k, and $j \succsim_i k$ if $j \succ_i k$ or $j \sim_i k$. The *primitives* of any one-to-one matching problem are thus L, N and P. A marriage problem with location restrictions is a roommate problem (L, N, P) where N is the union of two disjoint finite sets: a set of men $M = \{m_1, \ldots, m_h\}$, and a set of women, $W = \{w_1, \ldots, w_f\}$, where possibly $h \neq f$, and P is a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), \ldots, P(m_h), P(w_1), \ldots, P(w_f)\}$. That is, each man (woman) prefers being unmatched to be matched with any other agent in M (W, respectively). Since the a marriage problem is a special kind of roommate problem we will, throughout the paper, use the more general set up and notation of the roommate problem, while sometimes specifically referring to the marriage problem whenever appropriate. A roommate problem is *individually rational* if all agents prefer to be matched rather than remain single: $\forall i, j \in N : j \succ_i i$.

2.2   Matching with location restrictions

We now formally introduce the idea that a match between two different agents must happen at a matching 'location'. Define the mapping $\lambda : N \to L \cup \emptyset$ to be a function that assigns a location to each player allowing for the possibility that agents are not assigned to any location $l \in L$:

Definition 1. $\lambda : N \to L \cup \emptyset$ is a **location mapping** when $\lambda(i) = \lambda(j) = l \in L$ and $i \neq j \Rightarrow \lambda(k) \neq l$ for all $k \in N \setminus \{i, j\}$

The mapping λ assigns a location to at most two agents and allows for agents not to be assigned to any location: $\lambda(i) = \emptyset$. Define $\lambda^{-1} : L \to N \cup \emptyset$ as the correspondence that assigns to each location l the set of agents that are located at this location. If no agent is located at $l \in L$, then $\lambda^{-1}(l) = \emptyset$. Let $\lambda \in \Lambda$ where Λ is the set of all possible location mappings. The concept of a location mapping naturally leads to the definition of a matching under location restrictions.

Definition 2. Given is $\lambda \in \Lambda$. A **matching** is a function $\mu_\lambda : N \to N$ satisfying the following properties:

1. $\forall i \in N, \mu_\lambda(\mu_\lambda(i)) = i$;

2. $\forall\ i \neq j : \mu_\lambda(i) = j \Leftrightarrow \lambda(i) = \lambda(j) = l \in L$.

Condition 1 implies that a matching must yield a partition of the set N into pairs and/or singletons. Condition 2 imposes that for two different agents to be matched they must be assigned the same location $l \in L$. Denote by $\mathcal{M}^*$ the set of all matchings. One interpretation of a matching problem with location restrictions is that in order for two agents to be matched at location l, they must both possess (be assigned) the matching right attached to matching location l, where l belongs to a limited set of possible matching locations. In other words, if an agent 'owns' a matching right to a certain matching location then no other agent can assign herself this matching right unless the 'owner' agrees to this.[4]

We assume that agents have no preference over the possible matching locations but only over their possible partners at such a location.[5] Agent $\mu_\lambda(i)$ is agent i's **partner** at μ_λ; i.e., the agent with whom she is matched to (possibly herself). A matching μ_λ is **individually rational** if each agent is acceptable to his or her partner, i.e. $\mu_\lambda(i) \succsim_i i$ for all $i \in N$. For a given matching μ_λ, a pair $\{i, j\}$ (possibly $i = j$) is said to form a **direct blocking pair** if they are not matched to one another but prefer one another to their partner at μ_λ *and* they can assign themselves to some location $l \in L$ to which no one else is assigned by the current location mapping λ, i.e. $j \succ_i \mu_\lambda(i), i \succ_j \mu_\lambda(j)$ and $\exists l \in L$ such that $\forall k \notin \{i, j\} : \lambda(k) \neq l$. This definition is

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[4]This interpretation is in line with that of Alcalde (1995) but additionally introduces scarcity of matching locations.

[5]We do so to focus on the consequences of introducing scarcity in the set of possible matching locations.

different from the classic definition of a **blocking pair** (Gale Shapley, 1962) since it requires two agents who prefer to be matched together to their current match, to be able to guarantee themselves a location to do so. We extend each agent's preference over her potential partners to the set of matchings in the following way. We say that agent i prefers $\mu'_{\lambda'}$ to μ_λ, if and only if agent i prefers her partner at $\mu'_{\lambda'}$ to her partner at μ_λ, $\mu'_{\lambda'}(i) \succ_i \mu_\lambda(i)$. Abusing notation, we write this as $\mu'_{\lambda'} \succ_i \mu_\lambda$. A coalition S is a subset of N. In order to study stability in this setting one needs to know how a given matching μ_λ can be altered by a coalition of agents $S \subset N$. We will introduce two different notions of enforceability: **direct enforceability** and **exchange* enforceability**. A specific stability concept is then defined using a given enforceability concept.

3   Enforceability, dominance and stability

3.1   Direct enforceability, dominance and stability

Define for any coalition S the set $L_\lambda(S)$ as those locations that members of coalition S can, on their own, control: no one outside of S is assigned to a location in $L_\lambda(S)$:

$$L_\lambda(S) = \left\{ l \in L, \lambda^{-1}(l) \subset S \right\}$$

Note in particular that this definition implies that all $l \in L$ such that $\lambda(l) = \emptyset$ belong to $L_\lambda(S)$: all unassigned locations can be controlled by the members of S when contemplating a deviation from the current matching. We can now state the definition of direct enforceability:

Definition 3. Given is a matching $\mu_\lambda \in \mathcal{M}^*$. A coalition $S \subseteq N$ is said to be able to directly enforce a matching $\mu'_{\lambda'}$ over μ_λ, denoted by $\mu_\lambda \to_S \mu'_{\lambda'}$, if the following conditions hold for any agent $i \in N$:

1. $\mu'_{\lambda'}(i) \notin \{\mu_\lambda(i), i\}$ implies $\{i, \mu'_{\lambda'}(i)\} \subseteq S$; and

2. $\mu'_{\lambda'}(i) = i \neq \mu_\lambda(i)$ implies a) $i \in S$ or b) $i \notin S$ and $\mu_\lambda(i) \in S$ such that $\lambda'(\mu_\lambda(i)) \in L_\lambda(S)$.

Intuitively, this definition says that in order to 'deviate' to a new matching a set of agents S *cannot reassign* the locations of non-members without their permission. However, they *can reassign* the match of non-members by 'divorcing' from a non-member and taking up a different location, possibly by becoming single. For instance, if $j = \mu_\lambda(i)$ where $i \in S$ and $j \notin S$, then $\lambda'(i) \neq \lambda(i)$ implies that $\mu'_{\lambda'}(j) = \emptyset$. The definition simply means that if an agent obtains a new location right and by doing so becomes matched to a new partner at this location, if any, then it must be that both these agents agree to it and belong to S. In addition, such reallocation must be feasible among members of S only. Note that the above definition is written in terms of the matchings μ_λ and $\mu'_{\lambda'}$. It is insightful to rewrite the definition of direct enforceability in terms of the location assignment only.

Definition 4. Given is a matching $\mu_\lambda \in \mathcal{M}^*$. A coalition $S \subseteq N$ is said to be able to **directly enforce** a matching $\mu'_{\lambda'}$ over μ_λ, denoted by $\mu_\lambda \rightarrow_S \mu'_{\lambda'}$, if the following conditions hold for any agent $i \in N$:

1. $\lambda'(i) \neq \lambda(i) \Rightarrow i \in S$ and $\lambda'(i) \in L_\lambda(S)$;

2. $\lambda'(i) = \lambda(i)$ and $\lambda^{-1}(\lambda(i)) \neq \lambda'^{-1}(\lambda(i))$ implies $i \in S$ and $\lambda(i) \in L_\lambda(S)$.

Condition 1 implies that when an agent obtains a new location assignment, then this agent should belong to S and other members of S should be able to provide this agent with this new location from the set $L_\lambda(S)$: $\lambda'(i) \in L_\lambda(S)$. Condition 2 implies that when an agent accepts a new partner, while not having changed her own location assignment, then this agent must belong to S and the members of S control the other location right at that location or it is currently unassigned: $\lambda(i) \in L_\lambda(S)$. To illustrate how location scarcity affects direct enforceability consider the following example, adapted from Alcalde (1995):

Example 1. *Let* (L, N, P) *where* $L = \{l_1, l_2\}$, $N = \{1, 2, 3, 4\}$ *and* $P(1) = 2, 3, 4; P(2) = 3, 1, 4; P(3) = 1, 2, 4$ *and* $P(4) = 1, 2, 3$, *illustrated as follows:*

agent 1	agent 2	agent 3	agent 4
2	3	1	1
3	1	2	2
4	4	4	3

Consider a matching μ_λ with the following the location assignment $\lambda(1) = \lambda(3) = l_1$ and $\lambda(2) = \lambda(4) = l_2$ so that $\mu_\lambda = (13, 24)$. This means that agent 1 is matched to agent 3 at location l_1 and agent 2 is matched to agent 4 at location l_2. Then the set of agents $S = \{1, 4\}$ can not directly enforce a matching over μ_λ in which they are matched to one another. If the set of matching locations would be $L = \{l_1, l_2, l_3\}$ then they would be able to enforce such a matching.

The concept of enforceability does not depend on the preferences of the agents. Direct enforceability allows us to define the concepts of direct dominance and direct stability:

Definition 5. *Given is a matching problem (L, N, P).*

1. A matching μ_λ is **directly dominated** by $\mu'_{\lambda'}$ through coalition S, denoted by $\mu_\lambda <_S \mu'_{\lambda'}$, if there exists a coalition $S \subseteq N$ of agents such that $\mu'_{\lambda'} \succ_i \mu_\lambda$ $\forall i \in S$ and $\mu_\lambda \rightarrow_S \mu'_{\lambda'}$;

2. A matching μ_λ is **directly stable** if no other matching $\mu'_{\lambda'}$ directly dominates μ_λ.

Remark that if there exists a coalition $S \subset N$ such that $\mu_\lambda <_S \mu'_{\lambda'}$, then there also exist a pair of agents $\{i, j\} = S^* \subset S$, where possibly $i = j$, such that $\mu_\lambda <_{S^*} \mu^*_{\lambda^*}$. We say that such a pair $\{i, j\}$ is a **direct blocking pair**. Expressing direct dominance in terms of coalitions or in terms of pairs is thus equivalent. A matching μ_λ is *directly blocked by a coalition $S \subseteq N$* if there exists a matching $\mu'_{\lambda'}$ and a coalition S such that $\mu_\lambda <_S \mu'_{\lambda'}$. If S directly blocks μ_λ, then S is called a **direct blocking coalition** for μ. The **direct core** of a roommate problem with location restrictions, denoted by $C(L, N, P)$, consists of all matchings which are not directly blocked by any coalition. It is equivalent to the set of directly stable matchings. Using the notion of a direct blocking pair defined above we find that the definition of direct stability is equivalent to the non-existence of any direct blocking pair or individual: a matching μ_λ is directly stable if there does not exist a direct blocking pair $\{i, j\}$ where possibly $i = j$.

We now relate this concept to the classic stability concept introduced by Gale and Shapley (1962). According to Gale and Shapley a matching is stable if there does not

exist any **blocking pair or individual** in which a blocking pair is always allowed to deviate by getting matched. As illustrated by example 1, it is immediate that the concepts of direct stability and Gale Shapley stability are equivalent whenever $\mathcal{L} \geq \mathcal{N} - 1$. When this is the case then location restrictions are essentially immaterial and direct stability yields equivalent predictions as Gale Shapley stability. Two agents that would prefer to be linked could then always find a location that is currently assigned to no one else. When $\mathcal{L} < \mathcal{N} - 1$ agents have fewer opportunities to deviate and the possibility arises that a matching is directly stable while it is not stable in the Gale Shapley sense. The following example illustrates this:

Example 1 continued. *Since there exists an odd cycle $\{1, 2, 3\}$ this matching problem is 'unsolvable' in the Gale Shapley sense. The same conclusion is obtained when $L = \{l_1, ..., l_k\}$ where $k > 2$. However, when $L = \{l_1, l_2\}$, the following matchings are directly stable: $\mu_\lambda = (13, 24), \mu'_{\lambda'} = (14, 23), \mu^*_{\lambda^*} = (12, 34)$. Imposing more location restrictions leads to more matchings being directly stable. Consider (L, N, P) where $L = \{l_1\}$, then all (constrained)[6] Pareto optimal matchings $\{(12), (13), (14), (23), (24), (34)\}$ are directly stable.*

3.2   Exchange* enforceability, dominance and stability

Define for any coalition S the set $\mathcal{L}_\lambda(S)$ as the locations that members of coalition S *partially* control: any location in $L_\lambda(S)$ belongs to some member of S or does not belong to any agent according to location mapping λ :

$$\mathcal{L}_\lambda(S) = \{l \in L, \nexists i, j \notin S : \lambda(i) = \lambda(j) = l \}$$

$\mathcal{L}_\lambda(S)$ is the set of locations that the set of agents S can 'use' to form a new matching through exchanging location rights, respecting the location rights of agents who do not belong to S. Again, all locations that are not assigned under λ belong to this set, but now also those locations that *only* belong to one agent who does not belong to the set. We thus have that $L_\lambda(S) \subseteq \mathcal{L}_\lambda(S)$. Given $\mathcal{L}_\lambda(S)$ we can now state the definition of exchange* enforceability:

Definition 6. Given is a matching $\mu_\lambda \in \mathcal{M}^*$. A coalition $S \subseteq N$ is said to be able to **exchange* enforce** a matching $\mu'_{\lambda'}$ over μ_λ, denoted by $\mu_\lambda \rightleftharpoons_S \mu'_{\lambda'}$, if the following

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[6]Constrained in the sense that only one match can possibly be formed.

conditions hold for any agent $i \in N$:

1. $\mu'_{\lambda'}(i) \notin \{\mu_\lambda(i), i\}$ implies $\{i, \mu'_{\lambda'}(i)\} \cap S \neq \emptyset$ and $\lambda'(i) \in \mathcal{L}_\lambda(S)$;

2. $\mu'_{\lambda'}(i) = i \neq \mu_\lambda(i)$ implies $i \in S$ if $\lambda'(i) \neq \lambda(i)$.

Intuitively, this definition says that in order to enforce a new matching from the current matching, a set of agents S can do so by reshuffling the available location rights to members of S. The first condition implies that agents can create a new matching by exchanging their location rights among themselves. This can happen in two ways. First, an agent can obtain a new allocation right from another agent of the set S through a permutation of location rights among members of S. Second, the agent can obtain a location right that was not assigned to anyone under λ. The second condition allows agents to exchange their current location assignment (and thus their current partner) for a location that is not assigned to anyone else according to λ or by simply giving up their current assignment without being assigned a location according to λ'. It is natural to assume that when agents are allowed to exchange their locations, that then they should also be able to give up their current location or to exchange their current location for an available unassigned location, as long as no other agent of S assigns herself the same location. Similarly to the definition of direct enforceability we can rewrite the definition of exchange* enforceability as a function of the location mappings only.

Definition 7. Given is a matching $\mu_\lambda \in M$. A coalition $S \subseteq N$ is said to be able to exchange* enforce a matching $\mu'_{\lambda'}$ over μ_λ, denoted by $\mu_\lambda \rightleftharpoons_S \mu'_{\lambda'}$, if the following condition holds for any agent $i \in N$: $\lambda'(i) \neq \lambda(i) \Rightarrow i \in S$ and $\lambda'(i) \in \mathcal{L}_\lambda(S)$.

The key difference between direct and exchange* enforceability[7] is that the latter allows members of S to change the matching partner of agents who do not belong to S without their consent. For instance, if $j = \mu_\lambda(i)$ where $i \in S$ and $j \notin S$, then $\lambda'(i) \neq \lambda(i)$ implies that $\mu'_{\lambda'}(j) \in S \setminus \{i\}$ or $\mu'_{\lambda'}(j) = \emptyset$. The members of the set S cannot reassign the locations of non-members, but they can reassign the partner of non-members by exchanging their location right with some other agent or by simply exchanging their location for a currently available location. It follows that direct enforceability puts more restrictions on what a set of agents S can do in order to

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[7]Note again that the notion of exchange* enforceability does not depend on the preferences of the agents.

change a matching. We now show that when a set of agents can directly enforce a matching $\mu'_{\lambda'}$ over μ_λ then this set can also exchange* enforce $\mu'_{\lambda'}$ over μ_λ.

Lemma 1. *Suppose* $\mu_\lambda \rightarrow_S \mu'_{\lambda'}$ *then* $\mu_\lambda \rightleftharpoons_S \mu'_{\lambda'}$.

Proof. All proofs are in the Appendix.

$\square$

Exchange* enforceability allows us to define the concept of exchange* dominance:

Definition 8. A matching μ_λ is **exchange* dominated** by $\mu'_{\lambda'}$ by coalition S, denoted by $\mu_\lambda \lhd_S \mu'_{\lambda'}$, if there exists a coalition $S \subseteq N$ of agents such that $\mu'_{\lambda'} \succ_i \mu_\lambda$ $\forall i \in S$ and $\mu_\lambda \rightleftharpoons_S \mu'_{\lambda'}$.

When the coalition S is a pair $\{i,j\}$ (singleton $\{i\}$) then we call this an **exchange* blocking pair** (singleton). We say that $\{i,j\}$ or $\{i\}$ **exchange* blocks** μ_λ. This definition allows for an exchange* blocking pair to become matched by reshuffling the available location rights. In contrast to direct dominance, the existence of a coalition $S \subset N$ such that $\mu_\lambda \lhd_S \mu'_{\lambda'}$ does not (always) imply[8] that there also exists a set of agents $\{i,j\} = S^* \subset S$, where possibly $i = j$, such that $\mu_\lambda \lhd_{S^*} \mu^*_{\lambda*}$. We therefore have the following stability definitions based on exchange* dominance:

Definition 9. A matching μ_λ is **exchange* stable** if it is not exchange* dominated by any pair or individual. A matching μ_λ is **coalition-exchange* stable** if it is not exchange* dominated by any coalition S.

Our definition of exchange* blocking is broader than the notion of ***exchange blocking*** introduced by Alcalde (1995): a pair of agents $\{i,j\}$ is said to *exchange block* a matching when, only by swapping their current partners, they both improve upon their current match. Alcalde (1995) assumes that *only* 'partner swaps' can happen and does not consider 1) that agents may decide to become single by giving up their current matching right and 2) that there are, possibly, available matching locations that agents can use to become matched by exchanging their current

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[8]Coalition-exchange* stability implies exchange* stability but not vice versa (see for instance Manlove, 2013).

location right for such an available location right. The concept of exchange* enforceability takes the latter two possibilities into account. We now illustrate these concepts in example 1.

Example 1 continued. *In the example, the matchings* $\{(13, 24)\}$ *is exchange stable according to Alcalde (1995). We obtain that*

1. *when* $L = \{l_1\}$, *then all individually rational matchings* $\{(12), (13), (14), (23), (24), (34)\}$ *are exchange* stable. This set is equivalent to the set of directly stable matchings.*

2. *when* $L = \{l_1, l_2\}$, *then only matching* $\{(13, 24)\}$ *is exchange* stable.*

3. *when* $L = \{l_1, l_2, ..., l_K\}$ *where* $K > 2$, *then no matching is exchange* stable.*

The literature studied a stability notion which simultaneously requires exchange stability and Gale Shapley stability[9]: a matching is both (coalition-) exchange stable **and** Gale Shapley stable if there is *no exchange blocking pair (coalition)* **and** *no blocking pair*. Requiring both Gale Shapley stability, implicitly assuming no location scarcity, and exchange stability, assuming location scarcity, seems a somewhat peculiar assumption. We propose to consider the set of property rights (available locations) as a primitive of the model and let the stability concept be based on the enforceability rules that define how agents can transform a current matching into another one. In particular, can matching rights be obtained through an exchange of location rights or not? When they can (not), the concept of exchange* (direct) enforceability is appropriate.

3.3 Direct and coalition-exchange* stability: mutually dependent concepts

Example 1 is illustrative of an important conclusion we draw: when considering the set of matching locations as a primitive of the model, coalition-exchange* stability is a refinement of direct stability. This is in contrast to the conclusion made by Alcalde (1995) that Gale Shapley stability and exchange stability are mutually independent concepts. We point out that Alcalde's conclusion is based, implicitly and explicitly, on the assumption that the sets of available matching rights are different when testing exchange stability and Gale Shapley stability. Once we fix the

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[9]See Chechlarova and Manlove (2005), Irving (2008), McDermid et al. (2007), Manlove (2013).

set of matching locations, then coalition-exchange* stability implies direct stability. We now show formally that exchange* stability is a refinement of direct stability. We first show the following lemma:

Lemma 2. *When a matching μ_λ is directly blocked by a couple $\{i,j\}$, where possibly $i = j$, then it is also exchange* blocked by $\{i,j\}$.*

Lemma 2 can be generalized: direct dominance implies exchange* dominance. Whenever a matching μ_λ is directly dominated by some matching $\mu'_{\lambda'}$ it follows that μ_λ is also exchange* dominated by the same matching $\mu'_{\lambda'}$.

Proposition 1. *Given a one-to-one matching problem (N,L,P). Then $\mu_\lambda <_S \mu'_{\lambda'} \Rightarrow \mu_\lambda \lhd_S \mu'_{\lambda'}$.*

We return briefly to the relationship between (coalition-) exchange* stability and other stability concepts. We obtain that whenever there is no scarcity in locations - when $\mathcal{L} \geq \mathcal{N} - 1$ - then requiring the absence of exchange* blocking pairs is equivalent to simultaneously requiring the absence of exchange blocking pairs[10] (exchange stability à la Alcalde, 1995) and the absence of blocking pairs (stability à la Gale Shapley, 1962).

Proposition 2. *Given a one-to-one matching problem (L,N,P). When $\mathcal{L} \geq \mathcal{N} - 1$, then (coalition-) exchange* stability is equivalent to requiring both Gale Shapley stability and (coalition-) exchange stability.*

4 Characterizing coalition-exchange* stable matchings

4.1 A link between exchange* dominance and indirect dominance

We can now be more precise about the matchings that are (coalition-) exchange* stable. To do so consider the following example.
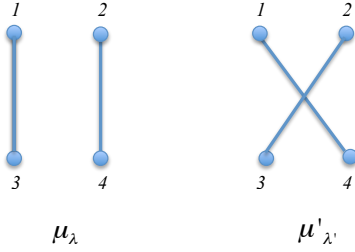
Example 2. *Let (L, N, P) where $L = \{l_1, l_2\}$, $N = \{1, 2, 3, 4\}$ and $P(1) = 3, 4, 1; P(2) = 4, 3, 2; P(3) = 2, 1, 3$ and $P(4) = 1, 2, 4$. Given these preferences this is equivalent to*

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[10]Note that coalition-exchange* stability is a refinement of exchange* stability.

a marriage problem and they are illustrated as follows:

agent 1	agent 2	agent 3	agent 4
3	4	2	1
4	3	1	2
1	2	3	4

Consider matchings $\mu_\lambda = (13, 24)$ and $\mu'_{\lambda'} = (14, 23)$, illustrated below:

These two matchings are directly stable (and also stable in the Gale Shapley sense). However, these matchings are not exchange* stable (nor exchange stable in the sense of Alcalde, 1995). Indeed, agents 3 and 4 can exchange* enforce matching $\mu'_{\lambda'}$ over matching μ_λ and agents 1 and 2 can exchange* enforce matching μ_λ over matching $\mu'_{\lambda'}$.

By close inspection we observe that while agents *1* and *2* cannot directly enforce matching μ_λ over matching $\mu'_{\lambda'}$, they may do so in two steps, assuming that both agents are forward looking. In a first step they could simply divorce in order to, in a second step, match with each other's ex-partners. Hence, if agents *1* and *2* cannot exchange their matching location rights, they may still be able to 'swap' their partners if they are not myopic. We generalize this intuition by showing that there is a close relationship between exchange* dominance and indirect dominance, a farsighted dominance concept introduced by Harsanyi (1974) and Chwe (1994) to study what happens when agents do not care about the immediate consequences of their actions but rather to the final outcome after other agents have reacted to their initial reaction. A matching $\mu'_{\lambda'}$ *indirectly dominates* μ_λ if $\mu'_{\lambda'}$ can replace μ_λ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching $\mu'_{\lambda'}$ compared to the status-quo. Formally, indirect dominance is defined as follows in our setting

Definition 10. A matching μ_λ is **indirectly dominated** by $\mu'_{\lambda'}$, denoted by $\mu_\lambda \ll \mu'_{\lambda'}$, if there exists a sequence of matchings $\mu^0_{\lambda^0}, \mu^1_{\lambda^1}, ..., \mu^K_{\lambda^K}$ (where $\mu^0_{\lambda^0} = \mu_\lambda$ and $\mu^K_{\lambda^K} = \mu'_{\lambda'}$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ such that for any $k \in \{1, ..., K\}$,

1. $\mu^K_{\lambda^K} \succ_i \mu^{k-1}_{\lambda^{k-1}} \; \forall i \in S^{k-1}$; and

2. coalition S^{k-1} can enforce the matching $\mu^k_{\lambda^k}$ over $\mu^K_{\lambda^K} \succ_i \mu^{k-1}_{\lambda^{k-1}}$.

Direct dominance can be obtained from definition 9 by setting $K = 1$. Obviously, if $\mu_\lambda < \mu'_{\lambda'}$ then $\mu_\lambda \ll \mu'_{\lambda'}$; i.e., direct dominance implies indirect dominance. The set of matchings that are not indirectly dominated by any other matching is the farsighted core:

Definition 11. Given is matching problem (L, N, P). A matching μ_λ belongs to the **farsighted core** (FC) if no other matching $\mu'_{\lambda'}$ indirectly dominates μ_λ.

When the set of locations is scarce, the farsighted core, if it exists, is not necessarily a singleton (see example 4). Indirect dominance offers the possibility to two agents to contemplate 'exchanging' their partners, as long as the latter would prefer to remain matched compared to being single. Exchange* dominance allows these two agents to do so directly, even if their partners would prefer to be single rather than being rematched through a swap they did not initiate. We now confirm this intuition in Theorem 1: whenever an individually rational matching exchange* dominates some other matching then it also indirectly dominates this matching.

Theorem 1. *Let* (L, N, P) *be a one-to-one matching problem with matching location restrictions. Let* $\mu'_{\lambda'}, \mu_\lambda \in \mathcal{M}^*$, *if* $\mu'_{\lambda'} \rhd \mu_\lambda$ *and if for all* i *such that* $\mu'_{\lambda'}(i) \neq \mu_\lambda(i)$ *it is that case that* $\mu'_{\lambda'}(i) \succcurlyeq_i i$ *then* $\mu'_{\lambda'} \gg \mu_\lambda$.

Theorem 1 implies that for individually rational matching problems the farsighted core must belong to the set of coalition-exchange* stable matchings. Example 3. shows that this result does not carry through to the case of matchings problems which are not individually rational: when $\mu'_{\lambda'} \rhd \mu_\lambda$ and $\mu'_{\lambda'}$ is not individually rational, then it is not necessarily the case that $\mu'_{\lambda'} \gg \mu_\lambda$.

Example 3. *Consider the following marriage problem* (L, M, W, P) *where* $M = \{m_1, m_2\}$ *and* $W = \{w_1, w_2\}$ *and* $L = \{l_1, l_2\}$ *with the following preferences:*

m_1	m_2	w_1	w_2
w_2	w_1	m_1	m_2
w_1	w_2	$\mathbf{w_1}$	$\mathbf{w_2}$
m_1	m_2	m_2	m_1

Let $\mu'_{\lambda'} = (m_1 w_2, m_2 w_1)$ *and let* $\mu_\lambda = (m_1 w_1, m_2 w_2)$, *as illustrated below:*

μ_λ $\mu'_{\lambda'}$
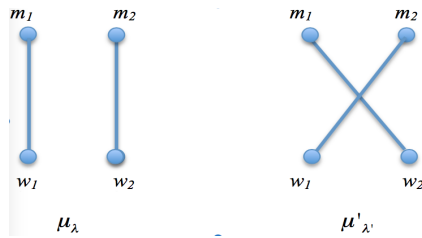
We then have that $\mu'_{\lambda'} \rhd \mu_\lambda$ *but not that* $\mu'_{\lambda'} \gg \mu_\lambda$. *Indeed, the women would never accept to remarry a different man than their partner in* μ_λ. *That is, exchange* enforceability can transform an individually rational match into an individually rational match. This possibility is ruled out by indirect dominance. Note that in this example the farsighted core is a singleton:* $FC = \{\mu_\lambda\}$ *while the set of coalition-exchange* stable matchings is empty.*

While Theorem 1 implies that for individually rational matching problems exchange* dominance entail indirect dominance, the converse is not the case: *indirect dominance does not imply exchange* dominance*. This is illustrated by example 4:

Example 4. *Consider the following marriage problem* (M, W, L, P) *where* $M = \{m_1, m_2\}$ *and* $W = \{w_1, w_2\}$ *and* $L = \{l_1, l_2\}$ *with the following preferences:*

m_1	m_2	w_1	w_2
w_2	w_2	m_1	m_1
w_1	w_1	m_2	m_2
m_1	m_2	w_1	w_2

Let $\mu'_{\lambda'} = (m_1 w_2, m_2 w_1)$ and let $\mu_\lambda = (m_1 w_1, m_2 w_2)$. We then have that $\mu'_{\lambda'} \gg \mu_\lambda$ but also that $\mu'_{\lambda'} \not\rhd \mu_\lambda$ since m_1 and w_2 cannot obtain the rights to a matching location at which they can match. In this example the farsighted core is a singleton: $FC = \{\mu_\lambda\}$ while the set of exchange* stable matchings is a couple: $\{\mu'_{\lambda'}, \mu_\lambda\}$.

Example 4 clarifies that for individually rational matching problems the farsighted core can be a strict subset of the set of exchange* stable matchings. We now show that this result depends on the level of location scarcity.

Definition 12. Given is matching problem (L, N, P). A matching μ_λ belongs to the **set of exchange* stable matchings** *(E*)* if a pair (or individual) of agents can enforce a matching that exchange* dominates μ_λ. A matching μ_λ belongs to the **set of coalition-exchange* stable matchings** *(C-E*)* if no other matching $\mu'_{\lambda'}$ exchange* dominates μ_λ.

When there is no scarcity ($\mathcal{L} \geq \mathcal{N} - 1$), then the set of coalition-exchange* stable matchings is equivalent to the farsighted core, while not necessarily equal to the set of exchange* stable matchings. We have the following corollary:

Corollary 1. *The farsighted core of any individually rational matching problem belongs to the set of exchange* stable matchings: if $FC \subset E^*$. However, $E^* \nsubseteq FC$. When $\mathcal{L} \geq \mathcal{N} - 1$, we have that $FC = C\text{-}E^* \subsetneq E^*$.*

4.2 Stable sets

Often times the farsighted core is empty which lead people to introduce alternative stability concepts to study farsightedly stable matchings. A popular stability concept is that of the **farsighted stable set** (FSS)[11]. A farsighted stable set of a matching problem is a set of matchings that satisfies ***internal stability*** - no matching of the set indirectly dominates another matching of the set - and ***external stability*** - all matchings outside the set are indirectly dominated by some matching(s) belonging to the set.

Definition 13. A set of matchings $V \subseteq \mathcal{M}^*$ is a **von Neumann Morgenstern farsighted stable set** (FSS) if

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[11]See Ray and Vohra (2014) for a recent analysis of the concept of farsighted stable set in coalition formation problems.

(i) for all $\mu_\lambda \in V$, there does not exist $\mu'_{\lambda'} \in V$ such that $\mu'_{\lambda'} \gg \mu_\lambda$ (internal stability);

(ii) for all $\mu'_{\lambda'} \notin V$ there exists $\mu_\lambda \in V$ such that $\mu_\lambda \gg \mu'_{\lambda'}$ (external stability).

In general, existence of a such a set is not guaranteed, nor its uniqueness when it exists. In the case of no scarcity, Mauleon et al. (2011) and Klaus et al. (2011) have shown that if a matching is GS stable (and thus directly stable in our setting), then it is a singleton FSS. When a matching is not stable, a FSS may not exist, as illustrated by our example 1 or may have more than two elements (see example 2 in Klaus et al. (2011).

Example 1 continued. *Let (L, N, P) where , $L = \{l_1, l_2, ..., l_k\}$ where $k > 2$, $N = \{1, 2, 3, 4\}$ and $P(1) = 3, 4, 1; P(2) = 4, 3, 2; P(3) = 2, 1, 3$ and $P(4) = 1, 2, 4$. Given these preferences this is equivalent to a marriage problem and they are illustrated as follows:*

agent 1	agent 2	agent 3	agent 4
2	3	1	1
3	1	2	2
4	4	4	3

It is easily verified that no FSS exists.

We define an **exchange* stable set** (ESS) as a set of matchings such that they do not exchange* dominate each other while any matching outside the set is exchange* dominated by some matching in the set.

Definition 14. A set of matchings $E \subseteq \mathcal{M}^*$ is an exchange* stable set (ESS) if

(i) for all $\mu_\lambda \in E$, there does not exist $\mu'_{\lambda'} \in E$ such that $\mu'_{\lambda'} \rhd \mu_\lambda$;

(ii) for all $\mu'_{\lambda'} \notin E$ there exists $\mu_\lambda \in E$ such that $\mu_\lambda \rhd \mu'_{\lambda'}$.

In the case when $\mathcal{L} \geq \mathcal{N} - 1$, Klaus et al. (2011) have shown that (lemma 1 in Klaus et al., 2011) any matching belonging to a FSS must be individually rational. Their result immediately extends to our setting with location restrictions.

Lemma 3. *Given is a matching problem (L, N, P). Let V be a FSS, then any $\mu_\lambda \in V$ is individually rational.*

We now show (proposition 3) that an ESS cannot be a strict subset of a FSS while the opposite can hold. The latter conclusion is demonstrated by example 4.

Proposition 3. *Given is a matching problem (L, N, P). Let V be a FSS and consider $V' \subsetneq V$, then V' cannot be a ESS.*

Example 4 continued. *Consider the following marriage problem (L, M, W, P) where $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$ and $L = \{l_1, ..., l_k\}$ where $k > 0$ with the following preferences:*

m_1	m_2	w_1	w_2
w_2	w_2	m_1	m_1
w_1	w_1	m_2	m_2
m_1	m_2	w_1	w_2

For all sets of location restrictions there is a unique FSS, and a unique ESS. For all $k > 1$ the FSS is a strict subset of the ESS.

1. *When $L = \{l_1\}$ we have that $FSS = ESS = \{(m_1w_1), (m_1w_2), (m_2w_1), (m_2w_2)\}$*

2. *When $L = \{l_1, ..., l_k\}$ we $k > 1$ have that $FSS \subsetneq ESS$. We have that $FSS = \{(m_1w_2, m_2w_1)\} \subsetneq \{(m_1w_2, m_2w_1), (m_1w_1, m_2w_2)\} = ESS$.*

We end our discussion of the exchange* stable set by providing an example which has no FSS but there exists a ESS.

Example 1 continued. *This matching problem does not have a FSS. Now consider the following matchings: $\mu_{\lambda^1}^1 = (13, 24)$, $\mu_{\lambda^2}^2 = (12, 34)$, and $\mu_{\lambda^3}^3 = (14, 23)$, in which agents are matched at locations l_1 and l_2 (all other locations are available), and let the set E be the set of all location permutations of these matchings (matched at l_i and l_j). There are $\binom{k}{2}$ matchings in E and E satisfies internal stability: no matching of E exchange* dominates another matching of this set. All other matchings are exchange* dominated by a matching of E (external stability). Hence, set E is an exchange* stable set.*

5 Computational complexity

In computer science the algorithmic aspects of matching problems have been studied at length. A large body of literature (see Gusfield and Irving, 1989 and Manlove, 2013) emerged studying whether deciding if a given roommate problem admits a (Gale Shapley) stable matching is a computationally complex question. A smaller literature[12] asked the same question while replacing Gale Shapley stability with the concept of exchange stability introduced by Alcalde (1995). In addition, Irving (2008) finds that deciding whether a roommate problem admits a matching which is simultaneously Gale Shapley stable and exchange stable à la Alcalde (1995) is computationally hard:

Theorem 2. *(Irving, 2008) "The problem of deciding whether a given stable roommate instance admits a stable matching that is exchange stable is NP-complete."*

Our proposition 2 shows that, when $\mathcal{L} \geq \mathcal{N} - 1$, coalition-exchange* stability is equivalent to simultaneously requiring Gale Shapley stability and coalition-exchange stability and corollary 1 shows that the set of coalition-exchange* stable matchings is equivalent to the farsighted core for an individually rational matching problem. We conclude that finding a farsightedly stable matching in an individually rational one-to-one matching problem without location scarcity is also computationally hard:

Corollary 2. *Let $\mathcal{L} \geq \mathcal{N} - 1$. Deciding whether an individually rational roommate problem admits a farsightedly stable matching is NP complete*

6 Conclusion

This paper contributes to the literature on the roommate and marriage problem in several dimensions. *First*, this paper explicitly introduces matching location restrictions in the one-to-one matching problem. When the set of matching locations is large (no scarcity), the matching problem is equivalent to the classic matching problem. When scarcity of matching locations exists some agents may want to form a new partnership without being able to implement it. *Second*, it develops the concepts of direct and coalition-exchange* stability in the general setting and

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[12]Cechlarova (2002), Cechlarova and Manlove (2005), Irving (2008), McDermid et al. (2007).

shows that coalition-exchange* stability is a refinement of direct stability. In the absence of location scarcity, direct stability is equivalent to Gale Shapley stability and exchange* stability is equivalent to simultaneously requiring exchange stability à la Alcalde (1995) and Gale Shapley stability. *Third*, the paper shows that there exists a natural relationship between indirect dominance and exchange* dominance allowing to conclude that set of coalition-exchange* stable matchings is a superset of the farsighted core, and equal to the farsighted core if locations are not scarce and the matching problem is individually rational. It is shown that an exchange* stable set can not be a strict subset of a farsighted stable set. In addition, an example is provided of a roommate problem in which no farsighted stable set exists while an exchange* stable set does exist. *Fourth*, by using well known complexity results, the paper obtains that deciding whether the farsighted core of an individually rational roommate problem exists is NP-complete.

Many questions remain unanswered. We have not fully characterized exchange* stable matchings. We have not tackled the question whether an exchange* stable set always exists. While we have shown that deciding whether the (unique) farsighted core of a individually rational roommate problem exists is computationally hard, we have not done so for individually irrational roommate problems. Nor have we discussed how to extend our setting to many-to-one or many-to-many matching problems. We leave these questions for future research.

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Appendix

Lemma 1. Suppose $\mu_\lambda \to_S \mu'_{\lambda'}$ then $\mu_\lambda \rightleftharpoons_S \mu'_{\lambda'}$.

Proof. Note that $\mu_\lambda \to_S \mu'_{\lambda'}$ implies that for all i such that $\lambda'(i) \neq \lambda(i) \Rightarrow i \in S$ and $\lambda'(i) \in L_\lambda(S)$. Since $L_\lambda(S) \subseteq \mathcal{L}_\lambda(S)$, let all these agents i belong to S'. But then, by the definition of exchange enforceability: $\mu_\lambda \rightleftharpoons_{S'} \mu'_{\lambda'}$ where we note that $S' = S$. ☐

Lemma 2. When a matching μ_λ is directly blocked by a couple $\{i, j\}$, where possibly $i = j$, then it is also exchange* blocked by $\{i, j\}$.

Proof. First assume that $i = j$. Hence for agent i we have that $i \succ \mu_\lambda(i)$. Then agent i can simply 'divorce' from $\mu_\lambda(i)$ without being assigned a new location. But the same move can be done through an exchange* blocking singleton: i just gives up her location assignment. Second assume that $i \neq j$ and $\mu'_{\lambda'}(i) = j$. Then it must be that $\lambda'(i) = \lambda'(j) = l'$ where $l' \in L_\lambda(\{i, j\})$. In other words, location $\lambda^{-1}(l') = \emptyset$: it was not assigned to anyone in λ. But then $\{i, j\}$ can exchange* enforce $\mu'_{\lambda'}$ over μ_λ, by exchanging their current location for l'. ☐

Proposition 1. Given a one-to-one matching problem (L, N, P). Then $\mu_\lambda <_S \mu'_{\lambda'} \Rightarrow \mu_\lambda \lhd_S \mu'_{\lambda'}$.

Proof. This proposition follows immediately from lemma 1. ☐

Proposition 2. Given a one-to-one matching problem (L, N, P). When $\mathcal{L} \geq \mathcal{N} - 1$, then (coalition-) exchange* stability is equivalent to requiring both Gale Shapley stability and (coalition-) exchange stability.

Proof. The proof is done for exchange* stability and exchange stability. Proving that it also holds for coalition-exchange* stability and coalition exchange stability follows the exact same lines and is therefore omitted.

$\Rightarrow$ Suppose that μ_λ is exchange* stable and there exists either a blocking pair (individual) or exchange blocking pair. Suppose first that $\{i, j\}$ is a blocking pair (or individual when $i = j$) of μ_λ. Since $\mathcal{L} \geq \mathcal{N} - 1$ $\{i, j\}$ can enforce the matching $\mu'_{\lambda'}$ where $\mu'_{\lambda'} = \mu_\lambda - i\mu_\lambda(i) - j\mu_\lambda(j) + ij$. We then have that $\mu'_{\lambda'} \rhd_{\{i,j\}} \mu_\lambda$, a contradiction. Now suppose that $\{i, j\}$ is a exchange blocking pair. But then for any $\mathcal{L}$ we have that $\mu'_{\lambda'} \rhd_{\{i,j\}} \mu_\lambda$, again a contradiction.

$\Leftarrow$ Suppose there does not exist a blocking pair, nor an exchange blocking pair but their exists a pair $\{i, j\}$ (or individual when $i = j$) and a matching $\mu'_{\lambda'}$ such that $\mu'_{\lambda'} \rhd_{\{i,j\}} \mu_\lambda$. Since $\{i, j\}$ is not an exchange blocking pair or individual(s), then it must be that they must be matched or alone in $\mu'_{\lambda'}$ in which they are better off. But then $\{i, j\}$ would be a blocking pair, a contradiction.

$\square$

Theorem 1. Let (L, N, P) be a one-to-one matching problem with matching location restrictions. Let $\mu'_{\lambda'}, \mu_\lambda \in \mathcal{M}^*$, if $\mu'_{\lambda'} \rhd \mu_\lambda$ and if for all i such that $\mu'_{\lambda'}(i) \neq \mu_\lambda(i)$ it is that case that $\mu'_{\lambda'}(i) \succcurlyeq_i i$ then $\mu'_{\lambda'} \gg \mu_\lambda$.

Proof. Let $B(\mu_\lambda, \mu'_{\lambda'})$ be the set of agents who are better off in $\mu'_{\lambda'}$ compared to μ_λ: $B(\mu_\lambda, \mu'_{\lambda'}) = \{i \in N, \mu_\lambda(i) \prec_i \mu_{\lambda'}(i)\}$. Let $I(\mu_\lambda)$ be the set of agents who are single in μ_λ: $I(\mu_\lambda) = \{i \in N, \mu_\lambda(i) = i\}$. We have that there exists a set of agents S who can exchange* enforce $\mu'_{\lambda'}$ over μ_λ and be better off in $\mu'_{\lambda'}$. We now construct an indirect dominance path from μ_λ to $\mu'_{\lambda'}$. Let $\mu^1_{\lambda^1}$ be a matching where all agents of S are single, if necessary by leaving their partner in μ_λ by giving up their location assignment under λ. Let $S_1 \subseteq S$ be those agents belonging to S who have a partner in μ_λ: $S_1 = S \setminus I(\mu_\lambda)$. We then have that $\mu_\lambda \rightarrow_{S_1} \mu^1_{\lambda^1}$ and for all $i \in S_1 : \mu'_{\lambda'} \succ_i \mu$. Now consider the set $S_2 = B(\mu^1_{\lambda^1}, \mu'_{\lambda'})$. For any $i \in S_2$ we have

1. $\mu'_{\lambda'}(i) \neq i$ and $\mu'_{\lambda'}(i) = j \in S$. Then it must be that $\lambda'(i) = \lambda'(j) = l' \in \mathcal{L}_\lambda(\{i, j\})$. But then $\lambda^{-1}(l') = \{i, j\}$ and hence $l' \in L_{\lambda^1}(\{i, j\})$; or

2. $\mu'_{\lambda'}(i) \neq i$ and $\mu'_{\lambda'}(i) = j \notin S$. Then it must be that $\lambda(j) = \lambda'(j) = l$ and $l \in \mathcal{L}_\lambda(S)$. But then $\lambda^{-1}(l') = \{i, j\}$ and hence $l' \in L_{\lambda^1}(\{i, j\})$; or

3. $\mu'_{\lambda'}(i) = i$. But then $i \notin S_2$.

We then have that for all $i \in S_2 : \lambda'(i) \in L_{\lambda^1}(S_2)$ and hence we have that $\mu^1_{\lambda^1} \rightarrow_{S_2} \mu'_{\lambda'}$. We conclude that $\mu'_{\lambda'} \gg \mu_\lambda$.

$\square$

Corollary 1. Given is an *individually rational* one-to-one matching problem (L, N, P). We have

1. $FC \subset E^*$. However, $E^* \nsubseteq FC$.

2. When $\mathcal{L} \geq \mathcal{N} - 1$, we have that $FC = C - E^* \subsetneq E^*$.

Proof. Given is that (L, N, P) is individually rational.

$\square$

1. Suppose first that $\mu_\lambda \in FC$ and $\mu_\lambda \notin E^*$. Then there exists $\mu'_{\lambda'}$ and $\{i, j\}$ where possibly $i = j$ such that $\mu'_{\lambda'} \rhd_{\{ij\}} \mu_\lambda$. Since $\mu'_{\lambda'}$ is individually rational, it follows from Theorem 1 that $\mu'_{\lambda'} \gg \mu_\lambda$, a contradiction. That $E^* \nsubseteq FC$ follows from example 4.

2. Now let $\mathcal{L} \geq \mathcal{N} - 1$, and let $\mu_\lambda \in C - E^*$. Suppose that there exists $\mu'_{\lambda'}$ such that $\mu'_{\lambda'} \gg \mu_\lambda$. Consider the set $B(\mu_\lambda, \mu'_{\lambda'})$, then $\mu'_{\lambda'} \ntriangleright_{B(\mu_\lambda, \mu'_{\lambda'})} \mu_\lambda$. However, since $\mathcal{L} \geq \mathcal{N} - 1$, there are always enough matching locations to let the members of $B(\mu_\lambda, \mu'_{\lambda'})$ enforce any partner swap and/or any direct blocking coalition since there are at least $\frac{1}{2} \sharp B(\mu_\lambda, \mu'_{\lambda'})$ locations available for members of $B(\mu_\lambda, \mu'_{\lambda'})$ who want to be matched to each other. Hence it must be that $\mu'_{\lambda'} \rhd_{B(\mu_\lambda, \mu'_{\lambda'})} \mu_\lambda$, a contradiction. Exmaple 4 has illustrated that $FC = C - E^* \subsetneq E^*$ even if $\mathcal{L} \geq \mathcal{N} - 1$.

Lemma 3. Given a one-to-one matching problem (L, N, P). Let V be a FSS, then any $\mu_\lambda \in V$ is individually rational.

Proof. The proof is equivalent to the proof of lemma 1 in Klaus et al. (2011) and therefore omitted.

$\square$

Proposition 3. Given a one-to-one matching problem (L, N, P). Let V be a FSS and consider $V' \subsetneq V$, then V' cannot be a ESS.

Proof. Let $V' \subset V$ where V is a FSS. Let $\mu_\lambda \in V$ while $\mu_\lambda \notin V'$. Then it must be that there exists $\mu'_{\lambda'} \in V'$ such that $\mu'_{\lambda'} \rhd \mu_\lambda$, but since $\mu'_{\lambda'}$ is individually rational (using Lemma 3) we know (using Theorem 1) that $\mu'_{\lambda'} \gg \mu_\lambda$, violating internal stability of V, hence V is not a FSS, a contradiction.

$\square$

Corollary 2. Deciding whether an individually rational roommate problem admits a farsightedly stable matching is NP complete.

Proof. This follows immediately from Proposition 2, Corollary 1 above and Theorem 2 in Irving (2008).

$\square$