Endogenous network formation in Tullock contests*

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Abstract

In many contests competing agents (politicians, firms, soccer teams, etc.) form cooperative alliances. However, when agents share valuable resources or information they increase not only their own value for the prize but also their rivals' valuations. Hence it is not obvious that competitors decide to cooperate. We study the endogenous formation of networks of cooperation in the Tullock contest. We find that the network formation process can act as a barrier to the entry to the contest insofar as there can exist pairwise stable group dominant networks that hamper the participation of some competitors. Moreover, we show that the total welfare can be maximized under pairwise stable dominant group networks rather than the complete one. Furthermore, it may happen that the player who is driven out is the one with the highest ex-ante valuation.

1 Introduction

In many situations of rivalry one observes that competitors are also involved in cooperative relationships. Von Hippel [9] identifies that U.S. steel minimill producers cooperate by informally trading technical know-how. Firms can also decide to share fixed costs of R&D, while they nonetheless are competing for a patent. On the demand side, firms can enhance their potential customer base by exchanging customer information, even thought they are competing to

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be the first to launch a new product. Soccer teams in the same league may decide to exchange information about players that jointly attract their interest in order to reduce uncertainty about the value of the player, but compete for the services of the very same player. The latter two examples share the feature that cooperative agreements between rivals have the potential to increase the (expected) value of winning the competition.

In many instances in economics and politics, competition takes form in contests. Rivals decide to spend resources in order to increase the probability of becoming the (only) winner of the contest. Indeed, competition in most of the examples above could be described adequately in terms of a contest. In these situations, competitors are usually confronted with the following dilemma. By forming a cooperative link with a competitor they increase the value of winning the competition, but they also increase the value their cooperating rival attaches to winning the competition. It is thus not obvious that rivals wish to form a link, even if there is any cost attached to doing it. Therefore, it is our aim to study the endogenous formation of cooperation networks in the classical model of contests: the Tullock model (Tullock, [8]). In order to do so, we employ a two stage model. In particular, in the first stage agents form a network of cooperation. We assume that the players are related symmetrically regarding the strategic value they obtain from a new link. In the second stage, agents take part in the Tullock contest.

We solve the model by backward induction and we rely on network theory from which we borrow the notion of pairwise stability (Jackson and Wolinsky, [3]). A (cooperative) network is pairwise stable if no agent wants to cut one of her links with other players and no two agents who are not linked find it in their interest to form a link.

Our main result is the characterization of all pairwise stable networks in the Tullock contest for small but negligible costs of link formation. We find that if a network is pairwise stable than it is either the complete network or it has a group dominant shape. The latter network architecture arises when there is a core of players who are fully linked with one another and all other agents have no links at all.

The existence of (group dominant) networks which are pairwise stable but different from the complete network is different from established results of network formation in the Tullock contest. Goyal and Joshi [1] consider a patent contest following Loury [4], which is, in the limit, a Tullock contest. They obtain that, when the cost of link formation is low, the only pairwise stable network is the complete network when the effort level in the patent contest is fixed and the same for all firms. In their terminology, the game has a "playing the field" structure: the expected marginal gross profit is only a function of the amount of links a player has and the aggregate amount of links of all other players. However, when agents can freely choose their effort intensity, the marginal gross benefits depend on the full distribution of the links of the others agents. We show that this feature allows for the existence of other pairwise stable networks. This idea has been studied in the related but different context of all pay auctions. In particular, Marinucci and Vergote, [5] have shown that when link formation affects the value of the prize in a multiplicative way, then group dominant networks and the complete network are the only pairwise stable networks. We thus confirm the results of Marinucci and Vergote [5] in the case of the Tullock contest: network formation can act as a barrier to entry to the Tullock contest. Nonetheless our results differ from Marinucci and Vergote [5] in three important ways. First, although we straightforwardly show that their results hold when the strategic value of link formation is multiplicative, we focus on the more difficult case where a link changes the value in an equal way for all agents. This assumption allows for a more natural interpretation of the effects of link formation. Second, we show that, by means of an example, there can be pairwise stable group dominant networks which yield a higher level of welfare than the complete network. Third we show, as a corollary to our results, that network formation has the potential to eliminate agents who have initially the highest valuation for to the prize of the contest.

We equally contribute to the theoretical literature on Tullock contests. Stein [7] has shown that when the valuation of a player increases and, as a result, no other player decided to leave the contest, then the expected payoff of that player will increase. We complete the result of Stein [7] by showing that it holds even in the case where some players decide not to participate in the contest in the event of an increase in the valuation of one player. In addition, we show that if the value of two agents increases with the same amount, the same conclusion is reached. A stronger result is show if the strategic value of link formation is multiplicative: if any given amount of players see their value increase by the same percentage, then they all see their expected payoff from the Tullock contest increase. All players not involved see their payoff decrees.

Furthermore, Matros [6] analyzes a Tullock contest where players may have different exogenously given valuations for the prize. He shows that if a player with a higher valuation joins the contest then players with lower valuations may leave the competition. On the contrary, in our model since a player's valuation is endogenously affected by the network of collaborations, it may be that a player with a priori higher valuation is not able to join the contest, because of the existing link formation. In this case, network formation acts as a deterrence to new efficient potential entrants in Tullock contest.

The paper is organized as follows. In Section 2 we present the network cooperation stage. In Section 3 we describe both Tullock contest and Stein's main results. In Section 4 we show that besides the complete network, there also exist pairwise stable dominant group networks. To get more insights we also analyze what happens when valuations take different specifications. Section 5 provides a welfare analysis and Section 6 concludes.

2 Network cooperation

We consider a finite set of ex-ante identical agents, $N = \{1, \ldots, n\}$ with $n > 1$. Relationships between agents are captured by the binary variables $g_{ij} \in \{0, 1\}$ which denote a relationships between agent i and j. In particular, if there exists a link between agents i and j then g_{ij} takes the value of 1, and of 0 otherwise. As a consequence, the set of agents and the relationships between them define a network g while the set of all possible networks is Γ. Let $N_i(g)$ be the set of agents that have a link with player i given some network g then $\eta_i(g) = |N_i(g)|$ represents the number of agents linked with i. To simplify notation, $g + ij$ means that the link g_{ij} is added to the network g while, $g - ij$ corresponds to the network g without the link g_{ij}. We say that there exists a path between agents i and j if either $g_{ij} = 1$ or if there exists a sequence of l distinct players $\{k_1, k_2, \ldots, k_l\}$ such that $g_{ik_1} = g_{k_1k_2} = \ldots = g_{k_lj} = 1$. Network $\hat{g}$ is said to be a component of network g if for all i, j, $i \neq j$ belonging to $\hat{g}$, the there exist a path between i and j and for $i \in \hat{g}$ and $j \in g$, if $g_{ij} = 1$ then $j \in \hat{g}$. These are all the basic features which allows us to describe some network structures we will refer later on. In particular,

- the complete network g^N is characterized by $\eta_i(g^N) = n - 1$ for all $i \in N$,
- the empty network g^0 is characterized by $\eta_i(g^0) = 0$ for all $i \in N$,
- the network g^D is characterized by a dominant group structure when the component $\hat{g}^D = \{i \in N, \eta_i(\hat{g}^D) > 0\} \subsetneq N$ is complete and all $j \notin \hat{g}^D$ have no links: $\eta_j(g^D) = 0$

A network game is a game where every agent $i \in N$ announces its intended link $s_{ij} \in \{0, 1\}$ which all other agents $j \neq i$. If i wants to make a link with j, then $s_{ij} = 1$ and $s_{ij} = 0$ otherwise. A strategy in the network game for agent i is given by $s_i = \{s_{ij}\}_{j \neq i}$, which is a $n-1$ vector which belongs to the set of all possible strategies of agent i, i.e. S_i. We then have that $g_{ij} = 1$ if $s_{ij} = 1 = s_{ji}$ and $g_{ij} = 0$ otherwise. A strategy profile $s = \{s_i, \ldots, s_n\}$ induces a network $g(s) \in \Gamma$. Once a network is formed, we assume that each agent pays a *negligible* but positive cost $c > 0$ per link formed. Given a strategy profile s, the payoff of agent i is given by

$$\Pi_i(s_i, s_{-i}) = \pi_i(g(s)) - c \times \eta_i(g(s))$$

where, $\pi_i(g(s))$ is the agent i expected gain to participate in the contest. Given this framework we look for the pairwise stable networks according to the definition proposed by Jackson and Wolinsky [3], namely

Definition 1 (Pairwise Stability) *A network g is pairwise stable (PWS) if the following two conditions holds:*

1. *if* $g_{ij} \in g \Rightarrow \Pi_i(g + ij) > \Pi_i(g)$ *and* $\Pi_j(g + ij) > \Pi_j(g)$
2. *if* $g_{ij} \notin g$ *and* $\Pi_i(g + ij) > \Pi_i(g) \Rightarrow \Pi_j(g + ij) < \Pi_j(g)$

Intuitively, the two conditions state that, starting form a network g, no one wants to delete a link and no pair of agents want to form a new link respectively.

3 Network formation in Tullock Contests

Given a network g, the value for the prize of agent i is denoted as $v_i(g)$ which is known to all agents. We relabel the agents such that

$$v_1 \geq v_2 \geq \ldots \geq v_n \tag{1}$$

We assume that the creation of a new link between agents i and j allows them to increase their value of the prize. As mentioned in the introduction, this could be due to the sharing of valuable information, to the sharing of fixed costs, etc. Furthermore, we assume that the benefit β of the link formation on each agents' valuation is symmetric. In other words, when two agents form a link they get exactly the same benefit form the collaboration i.e. for any network g such that i and j are not linked in g we have:[1]

$$v_i(g + ij) - v_i(g) = v_j(g + ij) - v_j(g) = \beta > 0 \tag{2}$$

Each agent i can exert an effort level $e_i \geq 0$ and agent i wins the prize with probability $p_i(g) = \frac{e_i}{\sum_{j=1}^{N} e_j}$. Given any network g we can derive the equilibrium

[1] We will discuss the implication of a asymmetric benefit in the paragraph 4.2.2.

effort level of each agent and, as a consequence, her equilibrium expected payoff. In particular, the agent chooses the effort level that maximizes her expected payoff:

$$\max_{e_i(g)} p_i(g)v_i(g) - e_i(g)$$

It is well known (see for instance Hillman and Riley [2]) that the number of participating agents is the largest integer κ such that

$$v_\kappa(g) > \frac{\kappa - 2}{S_{\kappa-1}} = \frac{\kappa - 2}{\kappa - 1} h_{\kappa-1}(g) \tag{3}$$

where $S_{\kappa-1} = \sum_{j=1}^{\kappa-1} \frac{1}{v_j(g)}$ and $h_{\kappa-1}(g)$ is the harmonic mean of the largest $\kappa - 1$ valuations. Since an agent i would join the contest if and only if condition (3) is satisfied, only κ players with the largest valuation will be active in the contest, whereas players with valuation between $\kappa + 1$ and N stay out. Following Stein [7] we can write down both the equilibrium strategies and the equilibrium payoff as follows

Proposition 1 (Stein, [7]) *Suppose that $v(g) = (v_1(g), \dots, v_n(g))$ such that condition (1) is satisfied. Then $e^*(g) = (e_1^*(g), \dots, e_n^*(g))$ is a Nash equilibrium of the Tullock contest in which agent i makes the following equilibrium effort:*

$$e_i^*(g) = \begin{cases} \frac{\kappa-1}{\kappa} h_\kappa \left(1 - \frac{1}{v_i(g)} \frac{\kappa-1}{\kappa} h_\kappa\right) & \text{if} \quad i = 1, \dots, \kappa \\ 0 & \text{if} \quad i = \kappa + 1, \dots, n \end{cases} \tag{4}$$

Additionally, if agent i participates to the contest, the probability that agent i wins the prize, becomes

$$p_i^*(g) = 1 - \frac{1}{v_i(g)} \frac{\kappa - 1}{\kappa} h_\kappa(g)$$

as a result her equilibrium payoff is given by

$$\Pi_i^* = v_i(g) \left(p_i^*(g)\right)^2 \tag{5}$$

Stein [7] has shown that both the probability of winning and the expected payoff are positively related with the player's valuation. Hence, we have the following

Lemma 1 (Stein, [7]) *Given the number of participants then: (i) an increase in v_i implies that p_i increases and p_j decreases for $i \neq j$ and (ii) an increase in v_i implies that the expected payoff of player i increases and the expected payoff of each of the other players will decrease.*

Given these building blocks, we can summarize the timing of the contest in two stages. In the first stage, the agents form their collaboration through their link formation strategy and the network g determines the valuations $v(g) = (v_1(g), \ldots, v_n(g))$ of all players. In the second stage, playing according to the above described strategies, each agent would decide either to join the competition or to stay outside the contest.

4 Equilibrium network formation

4.1 Characterization of pairwise stable networks

From Equation (2) it is straightforward to see that each agent valuation positively depends on her amount of links, namely

$$v_i(g) = \underline{v} + \beta \ \eta_i(g) \tag{6}$$

where $\underline{v} \geq 0$ is a minimum reward that i gets whenever she is competing alone. Even though we will work with this additive valuation specification, our main results still remain true with a multiplicative effect as we show in section 4.2.1. Now, we will look for the conditions that characterize pairwise stable networks. In particular, since each link has a smaller but positive cost, all players that i) do not participate to the contest and ii) are linked in g, have an incentive to cut their link. This lead us to the following lemma

Lemma 2 *If a player $i \notin \kappa$ and $\eta_i(g) > 0$ then g is not stable. A necessary condition for the network g to be pairwise stable is that all non participating firms have no links.*

Furthermore, two players want to form a link if and only if it is beneficial for both of them, i.e. $\Pi_i^*(g + ij) > \Pi_i^*(g)$ and $\Pi_j^*(g + ij) > \Pi_j^*(g)$. At this point it is convenient to rewrite the equilibrium payoff stated in Equation (5) in a more explicit way:

$$\Pi_i^*(g) = v_i \left(1 - \frac{1}{v_i} \frac{(\kappa - 1)}{\kappa} h_k\right)^2, \qquad \forall \ i \in g. \tag{7}$$

When player i forms a link g_{ij}, the equilibrium payoff becomes

$$\Pi_i^*(g + ij) = (v_i + \beta) \left(1 - \frac{1}{v_i + \beta} \frac{(\kappa - 1)}{\kappa} h_k^{'}\right)^2 \tag{8}$$

where $h_k^{'} = \frac{\kappa}{S_k^{'}}$ and $S_k^{'} = S_k - \frac{1}{v_i} - \frac{1}{v_j} + \frac{1}{v_i+\beta} + \frac{1}{v_j+\beta}$. Comparing Equation (7) with Equation (8) we can see what happens when player i forms a new link. On the one hand, the valuation increases due to the new collaboration. On the other hand we have a change in probability of winning the contest. In particular, due to the higher valuation player i would exert a higher effort to win the contest, and this leads toward an increase in $p_i(g)$. Despite that, there is also an increase in the harmonic mean: player i's opponent has a higher valuation making her a fierce rival, and this leads toward a decrease in $p_i(g)$. A priori it is not possible to see the overall effect of a new link on the expected payoff. We will show that the positive effect dominates the negative only as $\kappa \geq 3$

Theorem 1 *Let g be such that $\forall\ i \notin \kappa$, i has no link. Let $\kappa \geq 3$ and consider two players i and j such that they both participate to the Tullock contest but they are not linked to one another in network g. In such a case agents i and j always form a link and participate to the contest in the network $g + ij$.*

Proof. See Appendix A.

We can deduce that if all agents participate to the contest then they are all linked and the complete network is pairwise stable. Nevertheless, in the next theorem we show our main result namely, g^N is not the only pairwise stable network.

Theorem 2 *When β is sufficiently high, there exist pairwise stable dominant group networks.*

Proof. See Appendix B.

In others words Theorem 2 says that all the agents engaged in the contest are linked among themselves whereas those who do not participate form no links. The finding corroborates the main result found in Marinucci and Vergote [5] which states that there exist pairwise stable dominant group networks in the all-pay auction setting. Furthermore, the following condition allows us to characterize all pairwise stable networks in the Tullock contest.

Theorem 3 *Take g^κ characterized by a κ-players dominant group. A necessary and sufficient condition for g^κ to be PWS is*

$$\frac{v_l + \beta}{h'_{-l}} < \frac{\kappa - 2}{\kappa - 1} \tag{9}$$

where $h'_{-l} = \frac{\kappa-1}{S'_{-l}}$, $S'_{-l} = S_{-l} + \frac{1}{v_z+\beta}$ and $l, z \notin \kappa$.

Proof. See Appendix C.

4.2 Robustness with respect to the valuation function

In this section we analyze different specifications of the valuation function given by Equation (6) and we wish to see whether our main result remains true. In particular, in what follows we analyze what happens when the benefit of a new link 1) has a multiplicative effect on the valuation, and 2) is not equally shared between the agents.

4.2.1 Multiplicative effect

We are interested to study if our results qualitatively change if we alter the way in which the strategic benefit from collaboration is shared among agents. Let us consider the following multiplicative valuation[2]

$$v_i = \phi^{\eta_i(g)} \qquad \text{with} \quad \phi > 1. \tag{10}$$

In this case the payoff of an agent i would still be given by Equation (7) whereas when player i forms a new link her payoff becomes

$$\Pi_i^*(g+ij) = (v_i\phi)\left(1 - \frac{1}{v_i\phi}\frac{(\kappa-1)}{\kappa}h_k^{'}\right)^2 \tag{11}$$

where $h_k^{'} = \frac{\kappa}{S_k^{'}}$ and $S_k^{'} = S_k - \frac{1}{v_i} - \frac{1}{v_j} + \frac{1}{v_i\phi} + \frac{1}{v_j\phi}$. In this case the result follows almost immediately from the multiplicative specification.

Theorem 4 *When the valuation is given by Equation (10) there exist PWS dominant group networks.*

Proof. See Appendix D.

4.2.2 Asymmetric benefit sharing

It is interesting to see what happens to the cooperation between agents if the benefit they get form the collaboration it is not equally shared among them. This could be the case if some agents have a higher bargaining power, or they are able to better exploit the information they get from cooperation. By means of an example, consider the valuation function of two agents i and j:

$$v_i = \underline{v} + \beta \qquad v_j = \underline{v} + \delta\beta \qquad \text{with } \delta \in (0,1)$$

It is not hard to see that if the asymmetry is small then because of the continuity of the payoff function in δ, our results would remain true. At the same time if the asymmetry is large enough, all room for cooperation disappears. We know from Lemma 1 that if v_i increases then both p_j and Π_j would decrease. Consequently, for values of δ low enough all the participating players would not being fully connected. As δ *tends* toward zero, player j would not wish to connect with player i and the dominant group network would no longer be pairwise stable.

[2] This specification is equal to the one used by Marinucci and Vergote [5] in an all-pay auction framework.

5 Welfare analysis

In this section we define total welfare as the sum of all agents expected payoffs and the network with the highest total welfare as the efficient one. Since some players do not participate and valuations are lower in group dominant networks compared to the complete network we may think that group dominant networks do in general not maximize total welfare. We show, however, by means of example, that it is not the case. Intuitively, when some agents do not participate, there is less competition, leading those who participate to the contest to spend less resources, yielding higher expected benefit for them but even higher total welfare for the networks as a whole. In particular, let us define the total welfare as $\omega(g^\kappa) = \sum_{i \in n} \Pi_i^*(g^\kappa) = \sum_{i \in \kappa} \Pi_i^*(g^\kappa)$, then

Theorem 5 *There exist values of β such that $\omega(g^\kappa) \geq \omega(g^N)$ with $\kappa < n$.*

Proof. The proof follows from the following example.

Example Six players want to participate in a contest. Consider first the network g^5 where one player, let say 1, does not participate, then the valuations of the competing players become

$$v_i = \underline{v} + 4\beta \qquad i = 2, \ldots, 6. \tag{12}$$

It is straightforward to see that the harmonic mean would be $h_5 = \underline{v} + 4\beta$ and consequently, the equilibrium payoffs are[3]

$$\Pi_1 = 0, \qquad \Pi_i = \frac{1}{25}(\underline{v} + 4\beta) \qquad i = 2, \ldots, 6.$$

accordingly, the total welfare is

$$\omega(g^5) = \frac{1}{5}(\underline{v} + 4\beta). \tag{13}$$

However, since $\Pi_i(g^5) \leq \Pi_i(g^5 + i1)$ for $i = 2, \ldots, 6$, then each player wants always to form the link g_{i1}. Consequently, g^5 is pairwise stable if and only if player 1 does not want enter the contest even when she form a link with a player $i \in g^5$. Hence the following condition must holds

$$(\underline{v} + \beta)\left(\frac{4}{\underline{v} + 4\beta} + \frac{1}{\underline{v} + 5\beta}\right) < 3 \qquad \Rightarrow 18\beta^2 - \underline{v}\beta - \underline{v}^2 > 0$$

this condition has only one positive root, i.e. $\beta^+ = \frac{v(1+\sqrt{73})}{36}$. Therefore g^5 is pairwise stable for any value of the benefit such that $\beta > \beta^+$. Now consider g^6, the valuations are

$$v_i = \underline{v} + 5\beta \qquad i = 1, \ldots, 6$$

[3]The incentive constraints for all participating player are satisfied whenever $\beta > 0$.

and the equilibrium payoffs become[4]

$$\Pi_i = \frac{1}{36}(\underline{v} + 5\beta) \qquad i = 1, \ldots, 6.$$

accordingly the total welfare is

$$\omega(g^6) = \frac{1}{6}(\underline{v} + 5\beta) \tag{14}$$

As a result, comparing Equation (13) and Equation (14) we get

$$\text{if} \quad \underline{v} \geq \beta > \beta^+ \quad \Rightarrow \quad \omega(g^5) \geq \omega(g^6)$$

Therefore there exists a parameters range such that a group dominant pairwise stable network maximizes the total welfare, but obviously the welfare distribution is less equally shared among the agents. However, despite this positive result the next corollary shows that the player with the originally larger valuation may be driven out of the contest through the network formation process.

Corollary 1 *The network formation process may lead to pairwise stable networks where the player with the initially higher valuation does not participate in the contest.*

Proof. The proof follows from the next example.

Example (cont.) Let us develop the previous example taking into consideration that player 1 has a higher ex ante valuation, namely

$$\underline{v}_1 = \underline{v} \cdot \epsilon \quad \text{with} \quad \epsilon > 1$$

as usual, the minimum reward for all the other players is $\underline{v}$. Let us analyze g^5 where player 1 does not participate. The valuation of all others players are still given by Equation (12) and as before, we know that each player i wants always to form the link g_{i1}, for $i = 2, \ldots, 6$. Again g^5 will be pairwise stable if and only if player 1 does not want enter the contest making a link with one competitor. So it must holds

$$(\underline{v} \cdot \epsilon + \beta)\left(\frac{4}{\underline{v} + 4\beta} + \frac{1}{\underline{v} + 5\beta}\right) < 3 \Rightarrow 36\beta^2 + \beta\underline{v}(22 - 24\epsilon) + \underline{v}^2(3 - 5\epsilon) > 0$$

we have just one positive roots, i.e. $\beta^+ = \frac{\underline{v}}{36}\left(12\epsilon - 11 + \sqrt{13 + 12\epsilon(12\epsilon - 7)}\right)$. Therefore the network g^5 is pairwise stable for $\beta > \beta^+$.

[4] The participation constrains are all satisfied for $\beta > 0$.

6 Conclusion

Starting from the observation that in many contests agents not only compete but also collaborate we have analyzed a two stage game where in the first stage agents choose their cooperation partners and in the second stage they compete in a Tullock contest.

In particular, we have showed that the network formation process may lead some agents not to enter in the competition stage. However the agents who take part in the contest are fully connected. This result does not seem to rely on the specific valuation functional form describing the benefit of cooperation as long as the latter is symmetric: both the additive and the multiplicative specification results in pairwise stable dominant group networks. This result is affected more by the hypothesis of symmetric benefit sharing of the cooperative gains. But, as soon an agent gets much more than the other one from the collaboration then it may be that in equilibrium the participating agents are not fully connected. Furthermore, even though the network formation process can act as barrier to the entry to the Tullock contest we showed that group dominant networks can be efficient. Thus, a higher level of total welfare can be reached even if some players do not participate to the competition. Unfortunately, it may happen that the player who is ruled out is the player with the highest ex-ante valuation.

A Proof of Theorem 1

In order to prove that is always beneficial for agent i to form a link with a competitor we must distinguish two cases. In particular, after that the link $g + ij$ is formed it may happen that 1) no agents drop out of the contest or 2) $k \geq 1$ agents drop out.

Case 1) We have to prove that the equilibrium payoff of player i (and j) is higher when she is linked with player j, namely

$$(v_i + \beta)\left(1 - \underbrace{\frac{(\kappa-1)}{S_k^{'}}\frac{1}{v_i+\beta}}_{a}\right)^2 > v_i\left(1 - \underbrace{\frac{(\kappa-1)}{S_k}\frac{1}{v_i}}_{b}\right)^2 \tag{15}$$

The previous condition holds whenever $a \leq b$. Therefore

$$a \leq b \Rightarrow S_k\beta + (v_i+\beta)\left(\frac{1}{v_i+\beta} + \frac{1}{v_j+\beta} - \frac{1}{v_i} - \frac{1}{v_j}\right) \geq 0 \tag{16}$$

From conditions given in Equation (4) it is easy to see that agent i participates in the contest if and only if her effort level is non negative. Hence, it must be that the participation constraints of both i and j are satisfied in network g: $v_i \geq \frac{\kappa-1}{S_k}$ and $v_j \geq \frac{\kappa-1}{S_k}$ which implies that

$$S_k \geq \frac{\kappa - 1}{\min(v_i, v_j)} \tag{17}$$

Let us analyze what happens when $v_i = \min(v_i, v_j)$. Replacing S with $\frac{\kappa-1}{v_i}$ in equation (16) and rearranging terms it is possible to rewrite it as

$$\left(\frac{\kappa - 2}{v_i}\right)\beta - (v_i + \beta)\left(\frac{1}{v_j} - \frac{1}{v_j + \beta}\right) \geq 0$$

since

$$v_j \geq v_i \Rightarrow \frac{v_i + \beta}{(v_j + \beta)v_j} \leq \frac{v_i + \beta}{(v_i + \beta)v_i} \leq \frac{1}{v_i}$$

we can rewrite the pervious condition as

$$\left(\frac{\kappa - 2}{v_i}\right)\beta - \frac{\beta}{v_i} \geq 0$$

which holds whenever $\kappa \geq 3$.
Now let us see what happens when $v_j = \min(v_i, v_j)$. Replacing S with $\frac{\kappa-1}{v_j}$ and repeating the same procedure we get

$$\left(\frac{\kappa - 2}{v_j}\right)\beta + \left(\frac{v_i + \beta}{v_j + \beta} - \frac{\beta}{v_i} - \frac{v_i}{v_j}\right) \geq 0$$

since

$$\frac{v_i^2 + v_j\beta + v_j(v_j - v_i)}{v_i v_j(v_j + \beta)} < \frac{v_i^2 + v_j\beta}{v_i v_j(v_j + \beta)} < \frac{v_i(v_i + \beta)}{v_i v_j(v_j + \beta)} = \frac{(v_i + \beta)}{v_j(v_j + \beta)}$$

we can rewrite the previous condition as

$$\frac{v_i + \beta}{v_j + \beta} < \kappa - 2 \Rightarrow v_i + \beta \leq (\kappa - 2)v_j + (\kappa - 2)\beta$$

this condition may be not satisfied when v_i is equal to its maximum value and v_j is equal to its minimum one,[5] namely

$$v_i = \underline{v} + (\kappa - 2)\beta \quad \text{and} \quad v_j = \underline{v}$$

however we can see what happens to condition (16) for these particular values. In this case, it holds $v_i \geq v_j$, then the condition on S_k becomes $S_k \geq \frac{\kappa-1}{\underline{v}}$. Replacing the new terms in condition (16) we get

$$\left(\frac{\kappa - 2}{\underline{v}}\right)\beta - \left(\frac{\underline{v}(\underline{v} + \beta) + (\underline{v} + (\kappa - 2)\beta)^2 - \underline{v}(\underline{v} + (\kappa - 2)\beta)}{\underline{v}(\underline{v} + \beta)(\underline{v} + (\kappa - 2)\beta)}\right)\beta \geq 0$$

which holds again whenever $\kappa \geq 3$.

[5]These values are found considering Lemma 2 holds true.

Case 2) Let us first analyze network g. We already know that for each player k it must hold $v_k > \frac{\kappa-1}{S}$, where κ is the number of players that take part to the contest and $S = \sum_{l=1}^{n} \frac{1}{v_l}$. From Lemma 1, we can immediately deduce that (i) a decrease in v_k implies both a decrease in $p_k(g)$ and an increase in $p_i(g)$ for $i \neq k$ and (ii) a decrease in v_k implies an increase of Π_i for all $i \neq k$.
Now, let us examine what happens to the network g when we perturbed it such that we change the value of k players so that they still participate in g. Specifically, take λ such that $v_k - \lambda \geq \frac{\kappa-1}{S}$ then it must hold $\Pi_i(g|perturbed) \geq \Pi_i(g)$. Hence, the $\lim_{v_k \to \frac{\kappa-1}{S}} \Pi_i(g|perturbed) = \Pi_i(g-k)$, with $g-k$ be the network where k players stay out. Therefore it must hold $\Pi_i(g-k) \geq \Pi_i(g)$.
Moreover if the link ij is formed such that both v_i and v_j increase then, following Case 1), their expected payoff increase, so that

$$\Pi_i(g-k+ij) \geq \Pi_i(g-k) \geq \Pi_i(g).$$

B Proof of Theorem 2

We already know by Theorem 1 that all participating agents are connected. Moreover, by Lemma 2 we know that all agents which do not participate to the contest are singletons. It remains only to show that there could be non participating agents. However, equation (3) implies that in equilibrium v_j would not to participate when

$$\frac{v_j}{h_{-j}} \leq \frac{\kappa-2}{\kappa-1}$$

In particular, when β is sufficiently high then the harmonic mean is high too making the participation of player j less certain. Moreover, when two rivals form a new link they increase h_{-j}, as a result the LHS would shrink while the RHS remains unchanged. Summing up, if β is high enough and the participating agents are sufficiently connected, it could be that j would not participate to the network $g+ij$ for all participating i.

C Proof of Theorem 3

Take any outsider player $l \notin \kappa$. In order not to participate in g^{κ}, l must not to violate the following conditions:

1. the participation constraint

$$\frac{v_l}{h_{-l}} \leq \frac{\kappa-2}{\kappa-1} \tag{18}$$

2. l has to remain out when she forms a link with $i \in g^{\kappa}$

$$\frac{v_l + \beta}{\bar{h}_{-l}} \leq \frac{\kappa - 2}{\kappa - 1} \tag{19}$$

where $\bar{h}_{-l} = \frac{\kappa-1}{\bar{S}_{-l}}$ and $\bar{S}_{-l} = S_{-l} - \frac{1}{v_i} + \frac{1}{v_i+\beta}$;

3. l has to remain out when she forms a link with $z \notin g^{\kappa}$

$$\frac{v_l + \beta}{h'_{-l}} \leq \frac{\kappa - 2}{\kappa - 1} \tag{20}$$

where $h'_{-l} = \frac{\kappa-1}{S'_{-l}}$ and $S'_{-l} = S_{-l} + \frac{1}{v_z+\beta}$.

Let us see which is the most restrictive condition. First, we can see if Equation (20) is more binding then Equation (18)

$$\frac{v_l + \beta}{h'_{-l}} \geq \frac{v_l}{h_{-l}} \Rightarrow (v_l + \beta)\left(S_{-l} + \frac{1}{v_z + \beta}\right) \geq v_l \;\; S_{-l}$$

which is always the case. Now, let us see if Equation (20) is more binding then Equation (19)

$$\frac{v_l + \beta}{h'_{-l}} \geq \frac{v_l + \beta}{\bar{h}_{-l}}$$

or

$$\left(S_{-l} + \frac{1}{v_z + \beta}\right) \geq \left(S_{-l} - \frac{1}{v_i} + \frac{1}{v_i + \beta}\right) \Rightarrow \frac{1}{v_z + \beta} + \frac{\beta}{v_i(v_i + \beta)} \geq 0$$

which always holds. We can conclude that the more binding condition is given by Equation (20).

D Proof of the Theorem 4

First of all we have to prove that if agents i and j are both participating in the contest then they alway want to cooperate and to form a link. Consequently we have to prove

$$(v_i\phi)\left(1 - \underbrace{\frac{(\kappa-1)}{S_k^{'}}\frac{1}{v_i\phi}}_{a}\right)^2 > v_i\left(1 - \underbrace{\frac{(\kappa-1)}{S_k}\frac{1}{v_i}}_{b}\right)^2$$

as before, it is enough to show that $a \leq b$ or that

$$\phi S_{\kappa}^{'} \geq S_{\kappa}$$

so, we have

$$\phi S_{\kappa}^{'} = \phi \left(\sum_{l \neq i,j} \frac{1}{v_l} + \frac{1}{\phi v_i} + \frac{1}{\phi v_j} \right) = \sum_{l \neq i,j} \frac{\phi}{v_l} + \frac{1}{v_i} + \frac{1}{v_j} \geq \sum_{l \neq i,j} \frac{1}{v_l} + \frac{1}{v_i} + \frac{1}{v_j} = S_{\kappa}$$

which is always true. Consequently, all the participating agents are always linked. Furthermore Lemma 2, stating that all non participating firms are not linked, extends to this case. It remains to show that there are such non participating players. It is straightforward to see that this is the case through the participation constraint given by Equation (3).

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