

On the Nature of Equilibria when Bertrand meets Edgeworth on Hotelling's Main Street*

Nicolas Boccard[†]

Xavier Wauthy[‡]

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Abstract

In this paper, we characterize the price equilibrium in a Hotelling model where firms face capacity constraints. For a wide range of capacity levels, a pure strategy equilibrium does not exist. When this is the case, we show that in a mixed strategy equilibrium firms use a finite number of prices and the number of price in their support differ by at most one.

1 Introduction

Bertrand (1883)–Edgeworth (1925) competition has generated a considerable amount of research in recent years. Curiously enough, most of this research effort remained confined to the case of markets where firms sell homogeneous goods. Building on the initial results of Kreps and Scheinkman (1983), the strategic value of capacity limitation as a way to relax price competition has been tested in various context. Some important references in this respect are Benoît and Krishna (1987), Davidson and Deneckere (1986), Deneckere and Kovenock (1992), Deneckere and Kovenock (2005) or Allen et al. (2000).

By contrast, almost no progress has been made in the analysis of Bertrand-Edgeworth competition in markets with differentiated products. A possible explanation for the lack of interest in Bertrand-Edgeworth competition with differentiated products is the fact that product differentiation already relaxes competition, so that firms enjoy positive payoffs in equilibrium. The corresponding models therefore offer tractable setups in which to study multi-stage games with various forms of commitment, which populates the IO field. Indeed, equilibrium outcomes in the corresponding price subgames are “appealing“. A second reason which possibly explains the relative disinterest for this class of games is the fact that the standard techniques which are used to solve Bertrand–Edgeworth games with homogenous goods do not work at all in the case of differentiated products. Apart from very special cases (such as Krishna (1989), Furth and Kovenock (1993) or Boccard and Wauthy (2005)), very little is known about the structure of mixed strategy equilibria in such markets. The

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[†]Departament d'Economia, FCEE, Universitat de Girona, 17071 Girona, Spain. Financial support from Generalitat de Catalunya (2005SGR213 and Xarxa de referència d'R+D+I en Economia i Polítiques Públiques) and Ministerio de Educación y Ciencia (SEJ2007-60671) are gratefully acknowledged.

[‡]CEREC, Facultés Saint-Louis Bruxelles and CORE, Université Catholique de Louvain.

aim of the present paper is precisely to offer a detailed characterization of the structure of these equilibria in a particular, though quite standard, model, namely, the Hotelling (1929) model.¹

More precisely, we shall analyze the structure of price equilibria in a pricing game where products are differentiated “à la Hotelling” and firms face rigid capacity constraints. It is well-known that in such a model, there will exist no pure strategy equilibrium when capacities take intermediate values, relative to the differentiation parameters. This result, though already conjectured by Edgeworth, has been proved formally in various contexts (Canoy (1996), Furth and Kovenock (1993), Wauthy (1996), Benassy (1989)). Yet, product differentiation and smooth rationing guarantee demand continuity which then translate to payoffs so that **the existence of a mixed strategy equilibrium is not an issue** (by Glicksberg (1952)’s theorem). What is at stake is the structure of a mixed equilibrium for such cases. In this contribution we establish essentially three results: first, firms use atoms in equilibrium, i.e. the support of the equilibrium mixed strategy is finite. Second, we show that the number of atoms used by the firms differ by at most 1. Third, the number of atoms in the support of an equilibrium decreases with the relative degree of product differentiation.

The present paper’s contribution is therefore mainly methodological. In particular we do not explicitly compute all the mixed strategy equilibria and do not characterize equilibrium payoffs. We are also silent on the uniqueness issue. All those issues are of course very relevant but we leave them to future research. Let us stress however that the methodology laid out in this paper is likely to be useful in addressing other economic problems characterized by continuous but non-concave payoffs. Typical examples are the class of switching-cost models with product differentiation (Klemperer (1987)) or duopoly models of price discrimination with differentiated products.

2 The model

2.1 Unconstrained Capacities

Two shops are located at the boundaries of the $[0; 1]$ segment along which consumers are uniformly distributed. The indirect utility derived by a consumer indexed by $x \in [0, 1]$ when buying product 1 or 2 is defined as $v - x - p_1$ and $v - (1 - x) - p_2$ respectively. Refraining from consuming any of the two products yields a nil level of utility. Consumers buy at most one unit of the product.

As in most of the literature, we assume market coverage in equilibrium i.e., that firms do not wish to act as local monopolists. This is guaranteed to happen if a monopolist covers the market. The monopoly profit being $p(v - p)$, the optimal price is $\frac{v}{2}$; it thus attracts the most distant customer if $v - 1 - \frac{v}{2} > 0 \Leftrightarrow v > 2$.

The *indifferent consumer*, denoted $\tilde{x}(p_1, p_2)$, solves $v - x - p_1 = v - (1 - x) - p_2$ i.e.,

$$\tilde{x}(p_1, p_2) = \frac{1 - p_1 + p_2}{2} \tag{1}$$

¹While we were in the process of rewriting this paper, our attention has been attracted on Sinitsyn (2009) who offers comparable characterizations in a different setup. While the basic games are very different, it is clear that the logic behind the characterizations are identical. Since the author does not mention any of our previous contributions in his paper, it seems fair to consider that his contributions and ours have been developed independently from each other.

The market is *non-covered* whenever

$$v - \tilde{x}(p_i, p_j) - p_i < 0 \Leftrightarrow p_i + p_j > 2v - 1 \quad (2)$$

i.e., at prevailing prices, at least one consumer prefers to refrain from buying. In such a case, firm i is a local monopolist with demand $D_i = \min\{1, v - p_i\}$. When firms name prices low enough, the market is *covered* so that demands are $D_1 = \tilde{x}(p_1, p_2)$ and $D_2 = 1 - \tilde{x}(p_1, p_2)$. Accordingly, the demand function of firm i is

$$D_i(p_i, p_j)_{|p_i \leq 2v-1-p_j} = \begin{cases} 0 & \text{if } p_i > p_j + 1 \\ \frac{1-p_i+p_j}{2} & \text{if } p_j - 1 \leq p_i \leq p_j + 1 \\ 1 & \text{if } p_i < p_j - 1 \end{cases} \quad (3)$$

$$D_i(p_i, p_j)_{|p_i \geq 2v-1-p_j} = \begin{cases} 0 & \text{if } p_i > v \\ v - p_i & \text{if } v - 1 \leq p_i \leq v \\ 1 & \text{if } p_i < v - 1 \end{cases} \quad (4)$$

In this section, each firm is able to cover the entire market so that profit is $\pi_i(p_i, p_j) = p_i D_i(p_i, p_j)$. The best reply is either $\max\left\{\frac{1+p_j}{2}, p_j - 1\right\}$ along (3) or $\max\{v - 1, 2v - 1 - p_j\}$ along (4) as $v > 2$ is assumed. The best reply is thus $\Phi(p_j) = \min\left\{\max\left\{\frac{1+p_j}{2}, p_j - 1\right\}, v - 1\right\}$. The standard Hotelling equilibrium $p_i = p_j = 1$ covers the market.

We analyze this game of complete information with the concept of Nash equilibrium. W.l.o.g. pure (price) strategies belong to the compact $[0; v]$. A mixed strategy is $\sigma \in \Delta$, the space of (Borel) probability measures over $[0; v]$, its support² is $\Gamma(\sigma)$, $\underline{\sigma} = \inf(\Gamma(\sigma))$ and $\bar{\sigma} = \sup(\Gamma(\sigma))$. The following result is well known but its proof includes unusual convexity considerations hinting at the results to come.

Lemma 1 *If $\min\{k_1, k_2\} \geq 1$, the equilibrium is uniquely defined by (1,1).*

Proof With ample capacities, sales equate demand. Since D_i is piecewise linear and decreasing in p_j , π_i is strictly concave in p_j whenever $D_i(p_i, p_j) > 0$. If positiveness was not an issue, we would be able to use the fact that the expectation of a strictly concave function is strictly concave to claim that the best response to a mixed strategy is a pure one because a strictly concave function has a unique maximum over a compact domain. As demand may become nil, the previous argument cannot be used. Yet, the knowledge of the best reply is useful to eliminate strictly dominated strategies.

It is trivial to check that no firm wants to price above $\sup_{p_j} \Phi(p_j) = v - 1$. Indeed, a lower price does better against any opponent's price, thus against any mixed strategy of the opponent. Using this information for the other player means that no one wants to price above $\Phi(v - 1) < v - 1$ (by construction of Φ). Reiterating, we see that no firm wants to price above the greatest fixed point of Φ . Likewise, no firm wants to price below $\inf_{p_j} \Phi(p_j) = \frac{1}{2}$. Hence, no firm wants to price below $\Phi_i(\frac{1}{2}) = \frac{3}{4}$. By the same token, no firm wants to price below the smallest fixed point of Φ . Since Φ has a unique fixed point at 1, uniqueness of the equilibrium is proved. ■

²Set of all points for which every open neighbourhood has positive measure.

2.2 Constrained Capacities

We now consider a pricing game $G(k_1, k_2)$ where (k_1, k_2) denote the capacity levels that firms are committed to. Firms cannot produce beyond capacities and thus ration consumers whenever demand exceeds capacity.³ In order to analyse the pricing game, we must therefore specify the organisation of the rationing. We follow Kreps and Scheinkman (1983) in assuming that the efficient rationing rule is at work in the market. Under this rule, rationed consumers are those exhibiting the lowest net reservation price for the good.

In the example depicted on Figure 1, all consumers located in $[0, \tilde{x}(p_1, p_2)]$ want to buy at firm 1 but some will be rationed. Under efficient rationing, those rationed consumers are located in $[k_1, \tilde{x}(p_1, p_2)]$. Note that they are precisely the most inclined to switch to firm 2 since they exhibit the largest net reservation price for product 2 within the set $[0, \tilde{x}(p_1, p_2)]$.

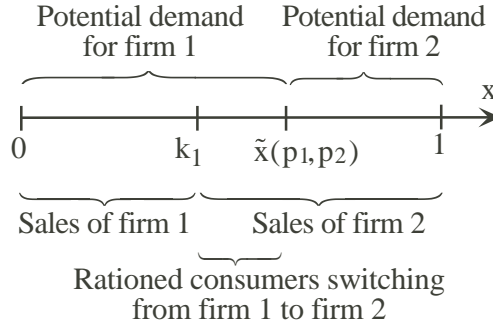


Figure 1: Demand vs. Sales

Extreme capacities, analyzed in Lemmas 1 and 7 (in the appendix), lead to the conclusion that firms have incentives to set intermediate values. Our focus is thus on the situation where each firm is unable to serve the entire market by herself but the industry can i.e., $k_2 \leq k_1 < 1$ and $k_1 + k_2 \geq 1$ (which implies $k_1 \geq \frac{1}{2}$).

3 Equilibrium analysis

3.1 Rationing

The original Hotelling analysis must be amended to account for binding capacities and rationing. Referring to equations (3–4), we may identify two different ways through which firm i 's demand, and therefore payoffs can be affected. For high prices (first segment), a firm may enjoy a positive demand if it captures part of the consumers rationed by firm j . More generally, in the domain where the market is covered, her demand may be strictly positive even for high price differentials. On the other hand, its sales are bounded from above by the installed capacity.

To derive sales, we must first delineate the areas where firms are constrained. According to the analysis developed when building equations (3) and (4), we already know that the price space must be partitioned according to market coverage. The solution in p_1 to equation (2) is

$$c(p_2) = 2v - 1 - p_2 \tag{5}$$

³The firm will refuse to produce beyond “capacity” if the marginal cost jumps beyond S .

We must also identify the constellations of prices for which capacity constraints are binding.

$$D_1(p_1, p_2) = k_1 \Rightarrow p_1 = a(k_1, p_2) \equiv p_2 + 1 - 2k_1 \quad (6)$$

$$D_2(p_1, p_2) = k_2 \Rightarrow p_1 = b(k_2, p_2) \equiv p_2 - 1 + 2k_2 \quad (7)$$

Hence, firm 1 is capacity constrained if $p_1 \leq a(k_1, p_2)$, no firm is capacity constrained if $a(k_1, p_2) \leq p_1 \leq b(k_2, p_2)$ and firm 2 is capacity constrained if $b(k_2, p_2) \leq p_1$. To connect areas, we solve:

$$c(p_2) = a(k_1, p_2) \Rightarrow p_2 = \rho(k_1) \equiv v + k_1 - 1 \quad (8)$$

$$c(p_2) = b(k_2, p_2) \Rightarrow p_2 = \delta(k_2) \equiv v - k_2 \quad (9)$$

Next, observe that $\delta(k_1) = a(k_1, \rho(k_1))$ and $\rho(k_2) = b(k_2, \delta(k_2))$. This means that the pair $(\rho(k_2), \delta(k_2))$ defines the highest prices such that firm 2 is exactly selling its capacity while firm 1 sells the market complement. A symmetric conclusion holds for $(\delta(k_1), \rho(k_1))$. In the region where the market is covered and firm 1 is capacity constrained ($a(\cdot) \geq p_1$), firm 2 sells $1 - k_1$ as long as $p_2 \leq \rho(k_1)$. The residual demand addressed to firm 2 is $1 - k_1$ in this region and is therefore locally independent of prices. Figure 2 illustrates our findings.

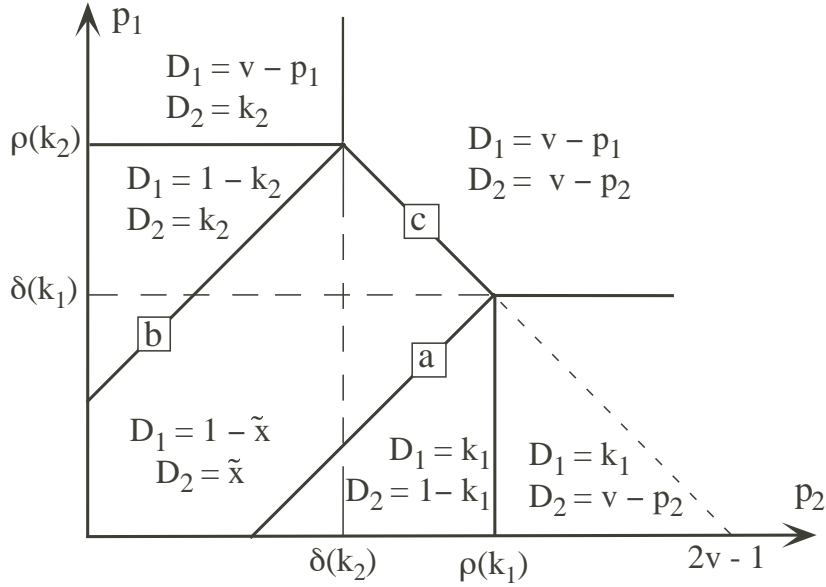


Figure 2: The Sales Space

Notice that $k_1 + k_2 > 1$ implies $a(\cdot) < b(\cdot)$ and $\delta(k_j) \leq \rho(k_i)$. The non void area delimited by the functions $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ is referred to as “the band”. We define $p_1^- = b(0, k_2)$ and $p_2^- = 2k_1 - 1 \leq 1$, the solution to $a(p_2, k_1) = 0$ ($k_1 \geq \frac{1}{2}$ implies $a(k_2, 0) \leq 0$).

Then the sales function of firm 1 can be defined as follows:

$$\text{if } p_2 \leq p_2^-, S_1(p_1, p_2) = \begin{cases} \tilde{x}(p_1, p_2) & \text{if } 0 \leq p_1 \leq b(p_2, k_2) \\ 1 - k_2 & \text{if } b(p_2, k_2) \leq p_1 \leq \rho(k_2) \\ v - p_1 & \text{if } \rho(k_2) \leq p_1 \leq v \end{cases} \quad (10)$$

$$\text{if } p_2^- \leq p_2 \leq \delta_2(k_2), S_1(p_1, p_2) = \begin{cases} k_1 & \text{if } 0 \leq p_1 \leq a(p_2, k_1) \\ \tilde{x}(p_1, p_2) & \text{if } a(p_2, k_1) \leq p_1 \leq b(p_2, k_2) \\ 1 - k_2 & \text{if } b(p_2, k_2) \leq p_1 \leq \rho(k_2) \\ v - p_1 & \text{if } \rho(k_2) \leq p_1 \leq v \end{cases} \quad (11)$$

$$\text{if } \delta_2(k_2) \leq p_2 \leq \rho(k_1), S_1(p_1, p_2) = \begin{cases} k_1 & \text{if } 0 \leq p_1 \leq a(p_2, k_1) \\ \tilde{x}(p_1, p_2) & \text{if } a(p_2, k_1) \leq p_1 \leq c(p_2) \\ v - p_1 & \text{if } c(p_2) \leq p_1 \leq v \end{cases} \quad (12)$$

$$\text{if } \rho(k_1) \leq p_2, S_1(p_1, p_2) = \begin{cases} k_1 & \text{if } 0 \leq p_1 \leq \delta(k_1) \\ v - p_1 & \text{if } p_1 \leq (k_1)\delta \end{cases} \quad (13)$$

The sales function $S_2(p_1, p_2)$ is entirely symmetric. Notice that sales are continuous and exhibit at most two constant segments corresponding to market shares k_i and $1 - k_j$. Glicksberg (1952) then ensures the existence of a Nash equilibrium.

3.2 Equilibrium Analysis

Having obtained the firms' sales function, we now characterize firms' best reply (BR) functions and identify Nash equilibria in $G(k_1, k_2)$. Given p_2 , we seek the local incentives for firm 1 with the help of on Figure 2.

Since $v > 2$, we have $\frac{v}{2} < \delta(k_1) = v - 1 + k_1$, hence the ideal monopoly price is below the line " $\rho(k_2) - c(\cdot) - \delta(k_1)$ " which means that the BR is at or below this line. Observe now that firm 1's demand is constant below the line " $a(\cdot) - \delta(k_1)$ ", hence the BR against $p_2 \geq \rho(k_1)$ is simply $\delta(k_1)$. Below $a(\cdot)$, demand is constant, hence the optimal price is $a(\cdot)$ itself and since it also belongs to the band, the BR is that of the band. The candidate in the band being $H_1(p_2) = \frac{1+p_2}{2}$, the BR inside the band is $\Phi_1(p_2) = \min \{b(k_2, p_2), \max \{H_1(p_2), a(k_1, p_2)\}\}$.

Above $b(\cdot)$, demand is constant, hence the optimal price is $\rho(k_2)$ which we may call a "security price" as it ensures the "minmax" payoff $\pi_i^s \equiv \rho(k_j)(1 - k_j)$. The *cut-off price* of firm 2 for which firm 1 is indifferent between $\Phi_1(p_2)$ and $\rho(k_2)$ is $\gamma(k_1, k_2)$ such that $\pi_1^s = \pi_1(\Phi_1(p_2), p_2)$. In Lemma 6 of the Appendix, we show uniqueness and derive the explicit form of $\gamma(k_1, k_2)$ as well as the conditions under which it is positive. Letting $\alpha(k_2)$ be defined as the solution to $a(k_1, p_2) = \frac{1+p_2}{2}$, we obtain $\alpha(k_1) = 4k_1 - 1$.

Firm i 's best reply correspondence, illustrated on Figure 3, is thus

$$BR_i(p_j) = \begin{cases} \rho_i(k_j) & \text{if } p_j \leq \gamma(k_i, k_j) \\ \frac{1+p_j}{2} & \text{if } \gamma(k_i, k_j) \leq p_j \leq \alpha(k_i) \\ p_j + 1 - 2k_i & \text{if } \max\{\alpha(k_i), \gamma(k_i, k_j)\} \leq p_j \end{cases} \quad (14)$$

The iterated elimination of dominated strategies along the lines of the proof of Lemma 1 leads to:

Lemma 2 $\Gamma(\sigma_i) \subset \left[\max \left\{ 1, \frac{\pi_i^s}{k_i} \right\}; \rho_i(k_j) \right]$.

Proof We proceed in several steps.

i) To show $\bar{\sigma}_i \leq \rho_i(k_j)$, observe that $\forall p_j \geq 0, \forall p_i \geq \rho_i(k_j), S_i(p_i, p_j) = v - p_i$, thus the average

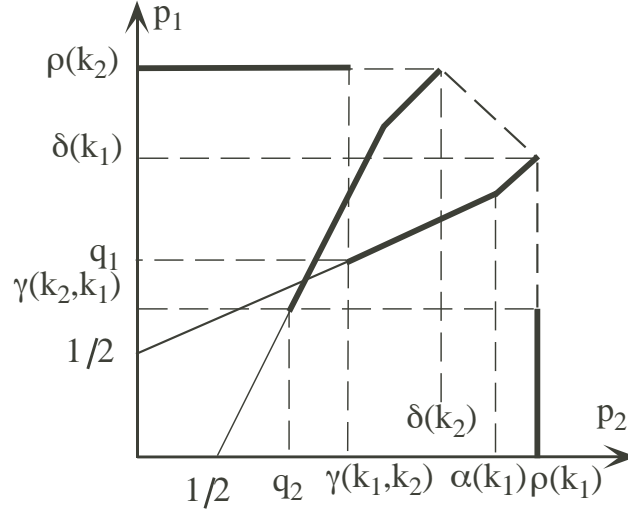


Figure 3: Demand vs. Sales

$\pi_i(\sigma_j, \cdot)$ is the monopoly payoff over that domain and it is strictly decreasing because the monopoly price $\frac{v}{2}$ is less than $\rho_i(k_j)$.

ii) Observe from (14), that BR_i is either strictly increasing or greater than unity and in any case $BR_i \geq H_i$ holds true. We may thus apply the proof of Lemma 1 to derive $\underline{\sigma}_i \geq 1$.

iii) The equilibrium payoff is larger than any deviation, hence $\pi_i^* \geq \pi_i(\sigma_j, \rho_i(k_j)) = \pi_i^s$. Together with the fact that $S_i \leq k_i$, we deduce $\underline{\sigma}_i \geq \frac{\pi_i^s}{k_i}$. ■

Proposition 1 *There exist only three possible type of equilibria:*

- i) *Both firms quote the pure strategy Hotelling price.*
- ii) *One firm plays a mixed strategy displaying $n + 1$ atoms while the other plays a mixed strategy displaying n atoms, with $n \geq 1$.*
- iii) *Both firms use a mixed strategy involving the same number of atoms.*

Proof Whenever $\gamma_1(k_1, k_2) \leq 1$ and $\gamma_2(k_1, k_2) \leq 1$, the best reply curves intersect at $(1, 1)$ meaning that the pure strategy price equilibrium $(1, 1)$ exists. Lemma 6 in the appendix proves that the condition reduces to $\min\{k_1, k_2\} \geq \frac{2-v+\sqrt{v^2-1/2}}{2} \in \left[\frac{1}{\sqrt{2}}, 1\right]$. The rest of the proof derives from 3 lemmas.

1. The support of a mixed strategy equilibrium distribution is finite.
2. The Hotelling equilibrium exists under sufficiently large capacities (case *i*).
3. Existence of an equilibrium where one firm uses a pure strategy while the other mixes over two atoms. This equilibrium exists for intermediate capacities (case *ii*).
4. Existence of $(n, n + 1)$ and $(n + 1, n + 1)$ equilibria, with $\infty > n \geq 1$ (case *ii*) continued and case *iii*).

Let $\beta_1(k_2) = 3 - 4k_2$ be the solution to $b(k_2, p_2) = \frac{1+p_2}{2}$. A similar argument can be used to identify $\beta_2(k_1)$ as the level of p_1 such that $a^{-1}(k_1, p_1) = \frac{1+p_1}{2}$ and $\alpha_2(k_1)$ as the level of p_1 such that $b^{-1}(k_1, p_1) = \frac{1+p_1}{2}$.

Lemma 3 *The support of a mixed strategy equilibrium is finite.*

Proof Using the fact that $\Gamma(\sigma_i) \subset [1; \rho_i(k_j)]$, we define $p_1^0 = p_2^0 = 1$ and for $m \geq 1$ let $p_1^m = b(p_2^{m-1}, k_2)$ and define p_2^m such that $p_1^m = a(p_2^m, k_1)$. Because industry capacity covers the market, those sequences are strictly increasing thus reach the upper bounds for a finite m .

Let p_1 vary in $[p_1^0; p_1^1]$. Observe that $\forall p_2 \geq p_2^0, D_1$ is decreasing concave (constant and then linear decreasing), thus so is π_1 . Since integration preserves this property, $\pi_1(\cdot, \sigma_2)$ has a unique maximum \underline{p}_1 over $[p_1^0; p_1^1]$ which is the unique price on which firm #1 may put mass in equilibrium.

Let p_2 vary in $[p_2^0; p_2^1]$. Conditional on $p_1 > p_1^1, D_2$ is decreasing concave, thus so is π_2 . The partial average $y_2(p_2) = \int_{p_1^1}^v \pi_2(p_2, x) d\sigma_1(x)$ has then a unique maximizer. Now, there is $\alpha_1 \geq 0$ such that $\pi_2(\cdot, \sigma_1) = \alpha_1 \pi_2(p_2, \underline{p}_1) + (1 - \alpha_1) y_2(p_2)$. Being the average of two locally concave functions, each with a unique maximizer, this payoff function has at most 2 maximizers over $[p_2^0; p_2^1]$. This shows that firm #2 uses at most 2 atoms over this interval.

$\forall \in [p_1^1; p_1^2], \forall p_2 > p_2^1, D_1$ is decreasing concave meaning that the partial average $y_1(p_1) = \int_{p_2^1}^v \pi_1(p_1, x) d\sigma_2(x)$ has then a unique maximizer. Since firm #2 uses at most 2 atoms over prices below $p_2^1, \pi_1(\cdot, \sigma_2)$ has at most 3 maximizers over $[p_1^1; p_1^2]$.

Repeating the argument for each player alternatively adds a single potential atom at each step and since all prices are bounded, the supports of an equilibrium distribution must be finite. ■

Lemma 4 *There exist mixed strategy equilibria where one firm uses a mixed strategy involving two atoms while the other uses a pure strategy⁴.*

Proof The proof is by construction. Firm 2 plays $\gamma_2(k_1, k_2)$ while firm 1 mixes $(\nu, 1 - \nu)$ over $\rho_1(k_2)$ and $\bar{p}_1 = \Phi_1(\gamma_2)$. The pure strategy of firm 2 guarantees that firm 1 is indifferent between her two atoms.

If the point (\bar{p}_1, γ_2) belongs to the border of the band then firm 2 has a constant demand for $p_2 > \gamma_2$ against $\rho_1(k_2)$ and \bar{p}_1 , thus a clear incentive to raise her price. The candidate equilibrium thus requires this point to be interior to the band i.e., $\bar{p}_1 = \frac{1 + \gamma_2(k_1, k_2)}{2}$, the Hotelling branch of the best reply function (14). The condition on the parameters is $\gamma_2 < \alpha(k_1)$ i.e., k_1 large relative to k_2 .

We now need to find weights that make $\gamma_2(k_1, k_2)$ a BR for Firm 2. We have

$$\pi_2(\nu, p_2) = \nu p_2 k_2 + (1 - \nu) p_2 \left(\frac{1 - p_2 + \bar{p}_1}{2} \right) \quad (15)$$

The FOC $\left. \frac{\partial \pi_2(\cdot)}{\partial p_2} \right|_{p_2 = \gamma_2(k_1, k_2)} = 0$ yields a solution ν^* since the first term is positive while the second is negative as $\gamma_2(k_1, k_2) < 1$. ■

Since we put no upper limit to ν , the above condition will be violated for ν large enough, thereby destroying our semi-mixed equilibrium. This is why we have to consider alternative candidate equilibria, involving both firms using mixed strategies. Let $(p_i^m)_{m=1}^{n_i}$ denote⁵ the (ordered) support of an equilibrium strategy σ_i . Given the matrix of price pairs $(p_1^m, p_2^s)_{\substack{m \leq n_1 \\ s \leq n_2}}$, we speak of a “line” when for a fixed p_1 and of a “column” for a fixed p_2 . Last, the “diagonal” is defined as the pairs $(p_1^m, p_2^m)_{m \geq 1}$.

⁴To the best of our knowledge, Krishna (1989) is the first paper identifying this particular type of equilibrium under Bertrand-Edgeworth competition.

⁵The “m” index bears to relation to that employed in Lemma 3.

Since the existence of an equilibrium is ensured by the continuity of payoff functions, if neither pure nor semi-mixed equilibria exist, then there exists an equilibrium in which both firms use non-degenerate mixed strategies. We furthermore know that firms use only atoms in an equilibrium. The intuition underlying the proof rests on the piecewise concavity of payoffs and on the fact that within the band, there can be only one best reply. Starting from the pair of minimal atoms, we identify some restrictions on the distribution of the other atoms; these restrictions allow us to show that the cases where firms use the same number of atoms or the case where the large capacity firm puts $n+1$ atoms while the other names n prices are the only possible cases that satisfy the necessary conditions for a mixed strategy equilibrium.

Lemma 5 *There exist completely mixed strategy equilibria where either $n_1 = n_2$ or $n_1 = n_2 + 1$.*

Proof We prove that the number of atoms used by firms differ by at most one and $\forall m \leq n - 1$, we have $1 - 2k_2 < p_1^m - p_2^m < 2k_1 - 1$. Observe firstly, as a by-product of Lemma 3, that $\forall p_i \in \Gamma(\sigma_i), \exists! p_j \in \Gamma(\sigma_j)$ such that (p_i, p_j) belong to the band; this is so because the profit in the band is strictly concave and has at most one maximizer (cf. y_2 function in the proof).

We use five successive claims with the help of Figure 4.

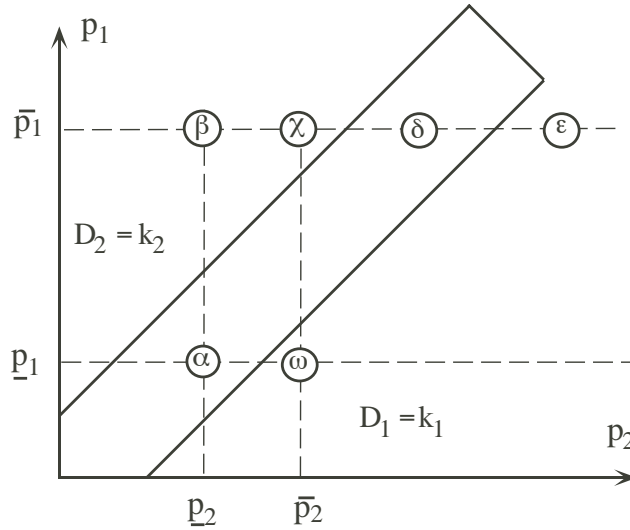


Figure 4: Atoms

1. The pair of minimal atoms (p_1^1, p_2^1) lies in the band.

If the minimal point was ω , not α , then firm #1 has a constant demand, thus would like to raise her price until ω hits the band. Likewise if the minimal point is β , firm #2 would like to raise her price until β hits the band. Define now $\alpha = (p_1^1, p_2^1), \omega = (p_1^1, p_2^2), \beta = (p_2^2, p_2^1)$.

2. ω and β lie strictly outside the band.

Point ω cannot belong to the band, for otherwise α would not be the unique maximizer of π_2 over $[0, k_1), b(p_1, k_2)]$ (recall that π_2 is concave over that domain given that #1 plays at or above p_1). The same applies for β wrt. profit maximization of firm #1.

3. The pair of minimal atom is strictly interior to the band.

If α lies on the right border of the band, then firm #2 has constant demand at p_2^1 because the

atoms of firm #1 are either above or below the band but not in. Price p_2^1 is then not a best reply to σ_1 . The same applies for the upper side of the band.

4. All points $(p_1^m, p_2^m)_{m \leq n-1}$ are strictly interior to the band.

The second “diagonal” point (p_1^2, p_2^2) could be either χ , δ or ε as shown on Figure A1. If χ , then the $\omega - \chi$ column would have no point in the band and $\pi_2(\sigma_1, \cdot)$ would be locally increasing at p_2^2 . Likewise, if ε , the $\beta - \varepsilon$ line would have no point in the band. Hence, $(p_1^2, p_2^2) = \delta$. We have thus shown that for $m \geq 1$, (p_1^m, p_2^m) lies in the band. Furthermore, the previous claim implies that the point is strictly interior if $m < \min\{n_1, n_2\}$. Only the last diagonal point could lie on one frontier of the band.

5. The number of atoms of equilibrium distributions differ by at most one.

Suppose $n_1 = n, n_2 = n + 1$. We first show using argument similar to above that the last diagonal point must be interior to the band. Then the last point has to belong to the small upper triangle for it the only way to generate an interior maximum of the profit function. The case for $n_2 > n + 1$ is now easy because $\pi_2(\sigma_1, \cdot)$ must have the same shape over $[p_2^n, p_2^{n+1}]$ or over $[p_2^{n+1}, p_2^{n+2}]$ leading to a contradiction.

■

Notice that, when establishing step 1 in the above proof, we have relied on the fact that the iteration process would reach its end when we reach the finite upper bound of the equilibrium support. At the same time, it appears from this proof that the number of possible iteration, i.e. the number of successive atoms, is actually depending on the width of the band, relative to the value of $\rho(k_i)$. Recalling that we have normalized the transportation cost to 1, v is sufficient to capture the relative importance of product differentiation into the model. An increase in v can be interpreted as a relative decrease in the degree of product differentiation. As a result an immediate by-product of the above argument is that the number of possible atoms in a mixed strategy equilibrium is inversely related to the magnitude of v . The less relevant is the differentiation dimension, the more numerous atoms are to be played in equilibrium.

4 Final Remarks

In this note, we have partially characterised the structure of mixed strategy equilibria in a Hotelling model where firms face capacity constraints. Our main results can be summarized as follows: in this setup, firms use a finite number of prices in equilibrium and the number of atoms they retain in equilibrium differ by at most one. This result is obviously a partial one and several extensions are called for. First, we do not fully characterize these equilibria. Second, we do not characterize, even partially, equilibrium payoffs. Third, we do not discuss uniqueness. These three problems should be addressed in order to allow for the analysis of capacity commitment games. This is to a large extent a matter of computations. Relying on preliminary results, we offer the following conjecture: despite of the multiplicity of equilibria (which is endemic in this class of models), there always exists an equilibrium in which the large capacity firm is held down to its minmax payoff.

Appendix

Lemma 6 *The cut-off price $\gamma(k_1, k_2)$ is defined as follows*

$$\gamma(k_1, k_2) = \begin{cases} \max \left\{ 0, \sqrt{8(v-1+k_2)(1-k_2)} - 1 \right\} & \text{if } k_2 \geq \frac{2-v+\sqrt{v^2-8k_1^2}}{2} \\ \frac{(v-1+k_2)(1-k_2)}{k_1} - 1 + 2k_1 & \text{if } k_2 \leq \frac{2-v+\sqrt{v^2-8k_1^2}}{2} \end{cases} \quad (16)$$

Proof The best reply of firm 1 within the band is either $H_1(p_2)$ or $a(k_1, p_2)$ depending on whether she hits her capacity or not. We have $D_1(H_1(p_2), p_2) < k_1 \Leftrightarrow p_2 \leq 4k_1 - 1$. Accordingly, the payoff accruing to firm 1 when she plays her best reply within the band is

$$\pi_1(p_2) = \begin{cases} \frac{\pi_1(p_2)}{8} = \frac{(1+p_2)^2}{8} & \text{if } p_2 \leq 4k_1 - 1 \\ \frac{\pi_1(p_2)}{8} = k_1(p_2 + 1 - k_2) & \text{if } p_2 \geq 4k_1 - 1 \end{cases} \quad (17)$$

These payoffs are then to be compared to those accruing from the security strategy $\rho_1(k_2)$, i.e. $\pi_1^s(k_2) = (v-1+k_2)(1-k_2)$. The solution to $\pi_1^s(k_2) = \pi_1(p_2)$ is $p_2 = \sqrt{8(v-1+k_2)(1-k_2)} - 1$ while that of $\pi_1^s(k_2) = \pi_1(p_2)$ is $p_2 = \frac{(v-1+k_2)(1-k_2)}{k_1} - 1 + 2k_1$. Observe that if v is too small, $\sqrt{8(v-1+k_2)(1-k_2)} - 1$ is negative and the security strategy is never used; also $\sqrt{8(v-1+k_2)(1-k_2)} - 1$ gives us a bound when considering large values of k_2 as $\pi_1^s(k_2)$ is decreasing. Taking care of the negative solution, we have

$$\sqrt{8(v-1+k_2)(1-k_2)} - 1 \leq 4k_1 - 1 \Leftrightarrow k_2 \geq \Phi(k_1) \equiv \frac{1}{2} \left(2 - v + \sqrt{v^2 - 8k_1^2} \right) \quad (18)$$

This separates cases where the security strategy beats the standard Hotelling best reply ($\pi_1(p_2)$) through the whole relevant range from the alternative case.

Hotelling Equilibrium The necessary conditions are $\gamma_1(k_1, k_2) < 1$ and $\gamma_2(k_1, k_2) < 1$. Observe from (6) that $\gamma_1(k_1, k_2) < 1 \Leftrightarrow$

$$\left\{ \begin{array}{l} k_2 \leq \Phi(k_1) \\ \frac{(v-1+k_2)(1-k_2)}{k_1} - 1 + 2k_1 < 1 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} k_2 \geq \Phi(k_1) \\ \sqrt{8(v-1+k_2)(1-k_2)} - 1 < 1 \end{array} \right\}$$

In the first case, the condition yield $k_2 > \frac{2-v+\sqrt{v^2-8k_1(1-k_1)}}{2}$. In the remaining case, the condition is $k_2 > \Phi(1/2) = \frac{2-v+\sqrt{v^2-1/2}}{2} \in \left[\frac{1}{\sqrt{2}}, 1 \right]$ (as $v > 2$). By symmetry for k_1 , the Hotelling equilibrium exists for capacities greater than $\Phi(1/2)$ (drawing the two curves shows that). ■

Lemma 7 *If $k_1 + k_2 < 1$, there is a unique equilibrium; it is in pure strategies.*

Proof When aggregate capacities are smaller than the market size ($k_1 + k_2 < 1$), firms do not compete at all since their potential market shares cannot overlap. In this “local monopoly” case, denoted $G^L(k_1, k_2)$, firms sell either at the monopoly price or at full capacity if the WTP is large enough. In any case, they have maximal incentives to increase capacities.

Demand is $D_i = \min \{v - p_i; k_i\}$ and profit is $\pi_i = p_i \min \{v - p_i; k_i\}$ which is maximum for $p_i = \max \{v - k_i; \frac{v}{2}\}$. Thus whenever $1 - k_j > k_i \geq \frac{v}{2}$, firm i acts as a pure monopolist whereas in the complementary case, she sells her capacity at the price $p_i^k = v - k_i$ which makes consumer located at a distance k_i indifferent between buying product i or refraining from consuming. ■

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