Exchange of indivisible goods and indifferences: the Top Trading Absorbing Sets mechanisms*

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Abstract

There is a wide range of economic problems involving the exchange of indivisible goods without monetary transfers, starting from the housing market model of the seminal paper of Shapley and Scarf [10] and including other problems like the kidney exchange or the school choice problems. For many of these models, the classical solution is the application of an algorithm/mechanism called Top Trading Cycles, attributed to David Gale, which satisfies good properties for the case of strict preferences. In this paper, we propose a family of mechanisms, called Top Trading Absorbing Sets mechanisms, that generalizes the Top Trading Cycles for the general case in which individuals can report indifferences, and preserves all its desirable properties.

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1 Introduction

Consider the economy modeled by Shapley and Scarf [10] in which there is a set of agents, each of whom has strict preferences over a set of indivisible goods, for instance houses. In this economy, commonly known as "housing market", each agent is endowed with one house and they are allowed to exchange their houses among themselves, although monetary transfers are not permitted. In this seminal paper, Shapley and Scarf prove the existence of a nonempty strict core in this economy by using a so-called Top Trading Cycles Algorithm (hereafter, TTC), attributed to David Gale.

The housing market has been extensively analyzed in the literature under the domain of strict preferences, becoming plain that the TTC mechanism satisfies (very) desirable properties. Roth and Postlewaite [8] prove that this mechanism results in the unique assignment which belongs to the strict core. Subsequently, Roth [7] shows that it is a dominant strategy for agents to reveal their true preferences. Furthermore, Ma [5] shows that the TTC mechanism, equivalent to the strict core mechanism, is the only mechanism satisfying individual rationality, Pareto-efficiency and strategy-proofness (in the domain of strict preferences).

Unlike the previous case, however, very few papers have been written on housing market under the full preference domain, even though it seems to be quite natural that agents may have indifferences over goods. One reason might be that introducing weak preferences to the model introduce also additional complications. First, in this case, the strict core might be empty, unique or multi-valued. Moreover, although the core is always nonempty, some of its allocations could be inefficient. As far as we know, there are two papers dealing with weak preferences. On the one hand, Quint and Wako [6] propose an algorithm to calculate if the strict core is empty or not, and obtain a strict core assignment if it is non-empty. However, this cannot be considered a mechanism since, for housing markets with an empty strict core, it reports that the strict core is empty but it does not give an allocation. On the other hand, Yilmaz [11] presents a random mechanism satisfying individual rationality, ex-ante efficiency and no justified-envy. However, this mechanism is not an strict core mechanism (i.e., there are housing market problems with non-empty strict core in which the mechanism do not select an strict core allocation). Additionally, it is not a generalization of the TTC mechanism, since the allocation that it proposes to problems with strict preferences may be different from the unique strict core allocation. Additionally, this mechanism does not satisfy strategy-proofness, although it attains higher levels of efficiency (by the randomized nature of the mechanism).

The contribution of this paper is to present a family of mechanisms that generalizes the TTC mechanism preserving their good properties when agents are allowed to report indifferences. In order to introduce this family of mechanisms, we define an algorithm called Top Trading Absorbing Sets algorithm (hereafter TTAS), which results in a strict core allocation when this set is non-empty and, otherwise, it results in a Pareto-efficient core allocation. Then, we prove that this family of mechanisms satisfy individual rationality, Pareto-efficiency and strategy-proofness.

Additionally, we have that other problems involving indivisible goods and where monetary transfers are not allowed have been considered so much in the literature. Some examples are the housing allocation with existing tenants (Abdulkadiroglu and Sonmez [1]), the kidney exchange problem (Roth et al. [9]) and the school choice problem (Abdulkadiroglu and Sonmez [2]). In these problems, the unique or one of the proposed solutions is based on an adaptation of the TTC to these frameworks. However, as in the housing market problem, they only study the case in which agents have strict preferences. Our family of mechanisms (with the same particular adaptations needed to each framework) will generalize all the classical mechanisms to the case in which agents can report indifferences.

The rest of the paper is organized into the following sections. Section 2 contains some basic preliminaries of the housing market problem. Section 3 revises the TTC mechanism, introduces the family of TTAS mechanisms and studies the properties of this family of mechanisms in the housing market problem. Section 4 presents some further applications of our mechanisms to other problems. Finally, an appendix contains the proofs of the results along the paper.

2 The housing market model

Let N be a finite set of agents and H be a set of houses such that $|N| = |H| = n$. Each individual $i \in N$ has a transitive and complete (but not necessarily antisymmetric) preference binary relation R_i on H. As usual, we will denote by P_i and I_i the symmetric part and the asymmetric part of R_i, respectively. For any R_i and any $S \subseteq H$, we will define the maximal elements of S according to R_i as the set $max(R_i) = \{x \in S \mid xR_iy \text{ for all } y \in S\}$. Define $R = (R_i)_{i \in N}$. Given $i \in N$, let $R_{-i} = (R_j)_{j \in N \setminus \{i\}}$ denote the preferences of all individuals except i.

An *assignment* (or *allocation*) is a bijective map $\mu : N \longrightarrow H$. In some cases, we will denote the house that is assigned to individual i by μ_i instead of $\mu(i)$. The assignment which describes the initial owners of the houses is called "initial endowment" and is denoted by ω. For any $T \subseteq N$, we define $\omega(T) = \{x \in H \mid x = \omega_i \text{ for some } i \in T\}$. Then, a housing market is a list (N, H, ω, R).

A *deterministic mechanism* f is a map that assigns for each housing market (N, H, ω, R) an assignment $f(N, H, \omega, R)$. When the description of (N, H, ω, R) is clear, we will denote the house assigned to individual $i \in N$ by the mechanism f as f_i. Let $\mathcal{F}$ be the set of all deterministic mechanisms. A *random mechanism* g is a probability distribution over $\mathcal{F}$. That is, a random mechanism associates for each housing market a probability distribution over the set of assignments. Obviously, any deterministic mechanism is a random mechanism.

An assignment μ is *individually rational* if for each agent $i \in N$, $\mu_i R_i \omega_i$. A deterministic mechanism f is *individually rational* if it always selects an individually rational assignment for each housing market. A random mechanism is *individually rational* if its support contains only individually rational deterministic mechanisms.

An assignment μ is *Pareto-efficient* if there does not exist any other assignment ν such that for all $i \in N$, $\nu_i R_i \mu_i$ and for some $j \in N$, $\nu_j P_j \mu_j$. A deterministic mechanism f is *efficient* if it always selects a Pareto-efficient assignment for each housing market. A random mechanism is *ex-post efficient* if its support contains only efficient deterministic mechanisms. A random mechanism g *stochastically dominates* other random mechanism h if for any possible vector of utilities $U = (u_i)_{i\in N}$ compatible with R, the following must hold: for all $i \in N$,

$$\sum_{x\in H} p(g_i(N,H,\omega,R) = x)\cdot u_i(x) \geq \sum_{x\in H} p(h_i(N,H,\omega,R) = x)\cdot u_i(x) \text{ and}$$
$$\text{there is some } j \in N \text{ in which this inequality is strict.}$$

Then, a random mechanism g is *ex-ante efficient* if it is not stochastically dominated by any other random mechanism.

A random mechanism g is *strategy-proof* if truth-telling is a dominant strategy in its associated preference revelation game. That is, for any possible vector of utilities $U = (u_i)_{i\in N}$ compatible with R, the following must hold: for all $i \in N$,

$$\sum_{x\in H} p(g_i(N,H,\omega,R) = x)\cdot u_i(x) \geq \sum_{x\in H} p(g_i(N,H,\omega,(R_{-i},R'_i)) = x)\cdot u_i(x)$$
$$\text{for all possible } R'_i$$

An assignment μ is in the *core* of the housing market if there is no coalition $T \subseteq N$ and matching ν such that for all $i \in T$, $\nu_i \in \omega(T)$ and $\nu_i P_i \mu_i$. An assignment μ is in the *strict core* of the housing market if there is no coalition $T \subseteq N$ and matching ν such that for all $i \in T$, $\nu_i \in \omega(T)$ and $\nu_i R_i \mu_i$ and for some $j \in T$, $\nu_j P_j \mu_j$.

Preliminaries in digraphs

We begin with some background in the theory of directed graphs. A *directed graph*, or digraph for short, is a pair (V,E), where V is a set of vertices (or nodes) and E is a set of directed arcs. The *indegree* (*outdegree*) of a node $v_i \in V$ is the number of arcs that point to (part from) v_i. Given two nodes $v_i, v_j \in V$, we say that there is a *path* from v_i to v_j if there is a sequence of nodes $v_i = v_1, \ldots, v_m = v_j$ such that for all $i \in \{1,..,m-1\}$, there is an arc from v_i to v_{i+1}. A *cycle* is an ordered set of nodes $C = \{v_1, v_2, ..., v_m\}$ such that for all $i \in \{1,\ldots,m-1\}$, there is an arc from v_i to v_{i+1} and there is an arc from v_m to v_1. Two nodes $v_i, v_j \in V$ constitute a *symmetric pair* if there is an arc from v_i to v_j and an arc from v_j to v_i.

An *absorbing set* is a set of nodes A that satisfies two conditions: (i) for any two nodes $v_i, v_j \in A$, there is a path from one to the other (inside connection), and (ii) there does not exist any path from any node $v_i \in A$ to any node $v_j \notin A$ (no inside-outside connection). An absorbing set is *paired-symmetric* if each of its nodes belongs to a symmetric pair.

3 Mechanisms

The classical framework in which the housing market problem is studied in the literature consists of individuals having strict preferences. In this case, Shapley and Scarf [10] have shown that the strict core always exists and have proposed the Strict Core mechanism, which selects for each housing market a strict core assignment.[1] It has been shown (Roth [7]) that this deterministic mechanism is strategy-proof. Moreover, Ma [5] shows that this is the unique mechanism that satisfies individual rationality, Pareto-efficiency and strategy-proof in this domain of strict preferences. Shapley and Scarf attributed to Gale an algorithm called Top Trading Cycles to compute the strict core assignment of a housing market.

Top Trading Cycles Mechanism

Consider a directed graph in which there are two types of nodes, agents and houses, arcs are formed by agents pointing to houses and houses pointing to agents, and all nodes have outdegree equal to 1. An interesting fact about any directed graph with these characteristics is that it always has at least one cycle and no two cycles intersect. This allows that the following algorithm, called Top Trading Cycles, attributed to David Gale and introduced by Shapley and Scarf [10], always determines an assignment.

Gale´s top trading cycles (TTC) algorithm:

Step 1:

(1.1) Let each agent point to her maximal house and each house point to its owner. Select the cycles of this graph.

[1] It has been proven by Roth and Postlewaite [8] that the strict core assignment is unique for housing markets with strict preferences.

(1.2) Their agents are removed from the algorithm by assigning each agent the house she is pointing to.

Step i:

(i.1) Let each remaining agent point to her maximal house among the remaining ones and each remaining house point to its owner (note that when an agent leaves, her original house also leaves; so a house remaining in the algorithm implies that her owner is still in the algorithm and vice versa). Select the cycles of this graph.

(i.2) Their agents are removed from the algorithm by assigning each agent the house she is pointing to.

In the general case in which we admit indifferences, the strict core may be empty. There is an algorithm (called *Top Trading Segmentation*) proposed by Quint and Wako [6] that determines if a housing market problem has an empty core or not and, if it is non-empty, it determines an allocation of it. Given that all the allocations of the strict core are indifferent for all the individuals (i.e., if μ and ρ belong to the strict core, $\mu_i I_i \rho_i$ for all $i \in N$), the case in which the strict core is non-empty has a good solution by this algorithm. However, there is not any satisfactory mechanism that serves for all housing market problems, independently if it has or not a non-empty strict core. Normally, the mechanism suggested in the literature to generalize the TTC mechanism (see the description for the case of indifferences in Shapley and Scarf [10] or Roth [7]) is the following:[2] (1) Take the preferences of the individuals who have indifferences and convert them in strict orders by some (fixed or random) tie-breakers; and (2) apply the Top Trading Cycles mechanism.

Obviously, this class of mechanisms coincides with TTC for the case of strict preferences. However, in the case of indifferences, the application of these mechanisms does not lead necessarily to efficient allocations. In fact, there are cases in which these mechanisms never achieve an efficient allocation, independently of the tie-breakers selected. We illustrate this with a simple example.

[2] Yilmaz [11] proposes other mechanism, but it is not a generalization of the TTC mechanism.

Example 1 *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $H = \{h_1, h_2, h_3, h_4, h_5\}$ *be the set of agents and houses. Let* $\omega_i = h_i$ *for all* $i \in N$ *be the initial endowment. The preference profile is the following:*

a_1	a_2	a_3	a_4	a_5
h_2	h_3	h_4, h_5	h_1	h_2
h_1	h_2	h_3	h_5	h_4
h_3	h_1	h_1	h_4	h_5
h_4	h_4	h_2	h_2	h_1
h_5	h_5		h_3	h_3

In this housing market problem, the strict core is empty and the core contains the following four allocations: $\mu^1 = (\mu^1_1, \mu^1_2, \mu^1_3, \mu^1_4, \mu^1_5) = (h_2, h_3, h_4, h_1, h_5)$, $\mu^2 = (\mu^2_1, \mu^2_2, \mu^2_3, \mu^2_4, \mu^2_5) = (h_1, h_3, h_5, h_4, h_2)$, $\mu^3 = (\mu^3_1, \mu^3_2, \mu^3_3, \mu^3_4, \mu^3_5) = (h_2, h_3, h_5, h_1, h_4)$ *and* $\mu^4 = (\mu^4_1, \mu^4_2, \mu^4_3, \mu^4_4, \mu^4_5) = (h_1, h_3, h_4, h_5, h_2)$. *There is only one indifference binary relation in the preference profile and, then, there are two possible results of the class of mechanisms presented before. They are exactly* μ^1 *and* μ^2. *However, it is easy to see that each of these allocations are Pareto dominated by* μ^3 *and* μ^4, *respectively.*

Then, we have that (i) on the one hand, the TTC mechanism behaves well for strict preferences, but the application of tie-breakers are not a good solution for the general case; and (ii) on the other hand, the Top Trading Segmentation algorithm provides a solution for some cases in the general case, but it is not a mechanism in the sense that it does not provide any allocation when the strict core is empty. In what follows, we propose a family of mechanisms (called Top Trading Absorbing Sets) for the general case which extends TTC and TTS and satisfies ex-post efficiency without renouncing to any of the good properties that TTC satisfies.[3]

Top Trading Absorbing Sets Mechanisms

For the introduction of the algorithm that determines the family of mechanisms presented below, we consider directed graphs in which there are two types of nodes, agents and houses, arcs are formed by agents pointing to houses and

[3]There could be other possible ex-post efficient mechanisms that always select core allocations (and strict core allocations if there are). However, many of them are not strategy-proof.

houses pointing to agents and all nodes have outdegree strictly positive. An interesting characteristic of these digraphs is that they always have at least one absorbing set (see Kalai and Schmeidler [4]).

Top trading absorbing sets (TTAS) algorithm:

Step 0: Consider a priority ranking of the houses; i. e., a complete, transitive and antisymmetric binary relation over H.

Step 1:

(1.1) Let each agent point to her maximal houses and each house point to its owner. Select the absorbing sets of this digraph.

(1.2) Consider the paired-simmetric absorbing sets. Their agents are removed from the algorithm by assigning them their current assignments (Obviously, these houses are removed too).

(1.3) Consider the remaining absorbing sets. Select for each agent a unique house to point to by using the following criterion: she point to the maximal house with the highest priority different from her current endowment.

(1.4) Then, in this subgraph, there is necessarily at least one cycle and no two cycles intersect. Assign (*temporarily*) to each agent in these cycles the house that she is pointing to, but maintain them in the algorithm.

Step i:

(i.1) Let each remaining agent point to her maximal houses among the remaining ones. Select the absorbing sets of this digraph.

(i.2) Consider the paired-simmetric absorbing sets. Their agents are removed from the algorithm by assigning them their current assignments (Obviously, these houses are removed too).

(i.3) Consider the remaining absorbing sets. Select for each agent a unique house to point to by using the following criterion: she point to the maximal house with the highest priority from those that has not been assigned to her yet.

(i.4) Then, in this subgraph, there is necessarily at least one cycle and no two cycles intersect. Assign (*temporarily*) to each agent in these cycles the house that she is pointing to, but maintain them in the algorithm.

The following example illustrates how the TTAS algorithm works for a particular housing market problem.

Example 2 *Consider a housing market with* $N = \{a_1, a_2, ..., a_9\}$ *and* $H = \{h_1, h_2, ..., h_9\}$ *and assume that the initial endowment of agent* a_i *is the house* h_i *for all* $i \in \{1, 2, \ldots, 9\}$. *Let the preference profile* R *be the following (we only express the houses that are not worse than the initial endowment of each agent):*

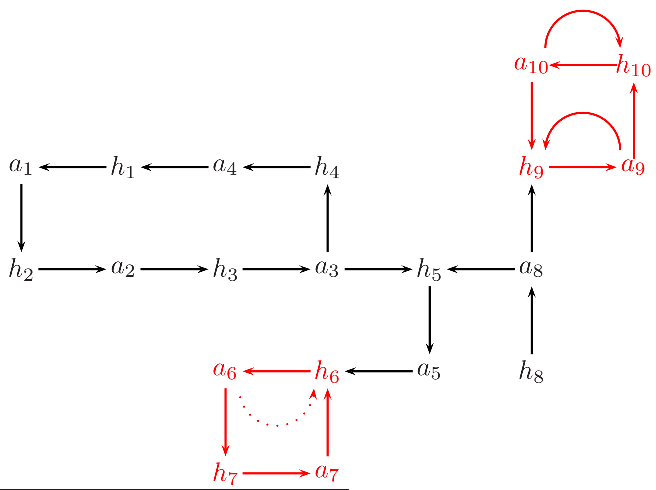
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
h_2	h_3	h_4, h_5	h_1	h_6	h_6, h_7	h_6	h_5, h_9	h_9, h_{10}	h_9, h_{10}
			h_5	h_2					
				h_4					
				h_5					

Consider the following priority ranking of houses:

$$h_1 \succ h_2 \succ h_3 \succ h_4 \succ h_5 \succ h_6 \succ h_7 \succ h_8 \succ h_9$$

In what follows we depict the directed graphs that are formed in each step of the algorithm: [4]

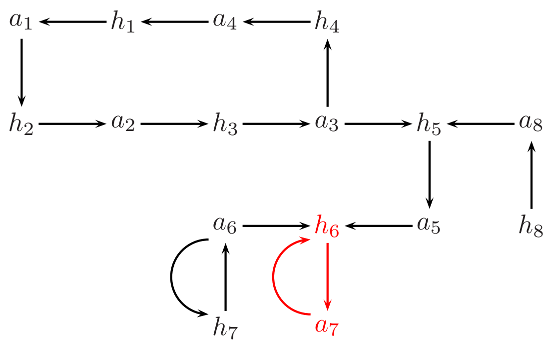
Step 1:



[4]There are two colors for the arrows in each graph: the black ones, that reflect the arrows that do not belong to an absorbing set; and the red ones, which are the ones that belong to an absorbing set. Moreover, within the set of red arrows, there are two types: the dotted arrows, which are the ones that in step (i.3) are not selected by the priority criterion $\succ$; and the normal ones, that represent the arrows chosen by the priority criterion.

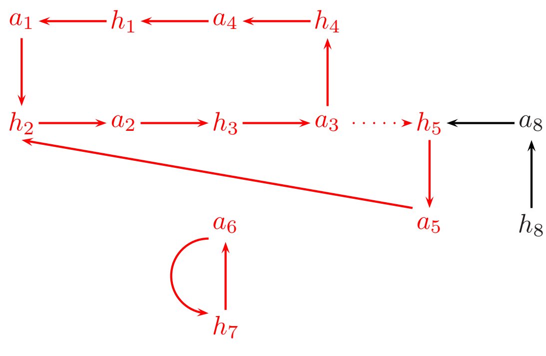
There are two absorbing sets: $A_1^* = \{a_9, h_9, a_{10}, h_{10}\}$, which is a paired-symmetric one and, hence, it is removed by assigning h_9 to $a9$ and h_{10} to a_{10}. The other absorbing set is $A_2 = \{a_7, h_7, a_6, h_6\}$. In this case, the priority ranking over houses is applied, and the cycle $c_2 = (a_6, h_7, a_7, h_6)$ is formed. Then, the algorithm assigns temporarily h_7 to a_6 and h_6 to a_7.

Step 2:



There is only one absorbing set: $A_3^* = \{h_6, a_7\}$, which is paired-symmetric. It is removed by assigning h_6 to a_7.

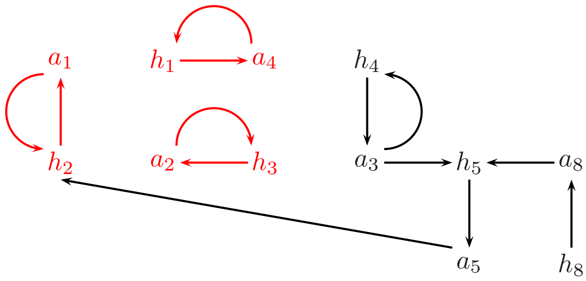
Step 3:



There is a paired-symmetric absorbing set $A_4^* = \{a_6, h_7\}$, which is removed by assigning h_7 to a_6. There is also another absorbing set $A_5 = \{a_1, h_1, a_2, h_2, a_3, h_3, a_4, h_4, a_5, h_5\}$. By applying the priority ranking, the cycle $c_5 = (a_1, h_2, a_2, h_3,$

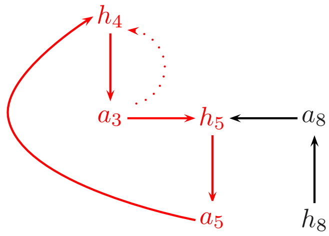
$a_3, h_4, a_4, h_1)$ *is formed. Then, the algorithm assigns temporarily* h_2 *to* a_1*,* h_3 *to* a_2*,* h_4 *to* a_3 *and* h_1 *to* a_4*.*

Step 4:



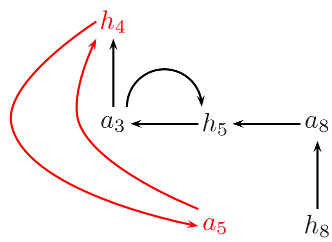
There are 3 paired-symmetric absorbing sets $A_6^* = \{a_1, h_2\}, A_7^* = \{a_2, h_3\}$ *and* $A_8^* = \{a_4, h_1\}$*, which are removed by assigning respectively* h_2 *to* a_1*,* h_3 *to* a_2 *and* h_1 *to* a_4*.*

Step 5:



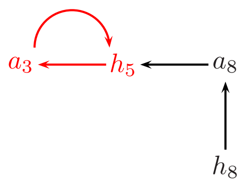
There is only one absorbing set $A_9^* = \{a_3, h_4, a_5, h_5\}$*. In this case, the cycle* $c_9 = (a_3, h_5, a_5, h_4)$ *is obtained by applying the priority ranking. Then, the algorithm assigns temporarily* h_5 *to* a_3 *and* h_4 *to* a_5*, respectively.*

Step 6:



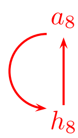
There is a paired-symmetric absorbing set $A^*_{10} = \{a_5, h_4\}$ *that the algorithm removes by assigning* h_4 *to* a_5*.*

Step 7:



There is a paired-symmetric absorbing set $A^*_{11} = \{a_3, h_5\}$ *that the algorithm removes by assigning* h_5 *to* a_3*.*

Step 8:



There is a paired-symmetric absorbing set $A^*_{11} = \{a_8, h_8\}$ *that the algorithm removes by assigning* h_8 *to* a_8*.*

Meanwhile in the description of the TTC is clear that the algorithm always determines an allocation, it is not so clear that this occurs with the TTAS. The following proposition shows it.

Proposition 1 *The TTAS algorithm always selects an allocation.*

It is easy to see that in the case in which all individuals have strict preferences, all absorbing sets that appear in any step i of the algorithm are cycles and, when the trading between the agents in each cycle is done, each of them forms a paired-symmetric absorbing set with her new house in step $i+1$ and leaves the algorithm. As a consequence, the TTAS coincides with the TTC when the preferences are strict.

The Top Trading Absorbing Sets Algorithm determines an allocation depending on the priority ranking $\succ$ selected in the Step 0. Then, we define a mechanism for each priority ranking in the following way: a mechanism μ is a Top Trading Absorbing Sets mechanism if there exists a priority ranking $\succ$ such that the mechanism chooses for each housing market problem the allocation that the Top Trading Absorbing Sets algorithm selects with this priority ranking. The selection of the priority ranking is important only in the case in which the strict core is empty, given that in the rest of cases it does not affect the welfare level that each individual attains. However, if the strict core is empty, the priority ranking indicates the individuals that, in case of conflict, have to be treated better than others (but always mantaining the efficiency of the mechanism and the fact that the allocation must be in the core of the problem). Although the priority ranking is written in terms of houses for the simplicity of the algorithm, its interpretation in terms of agents is easy: one individual i has priority over other j if the original house of i, $\omega(i)$, has priority over $\omega(j)$.

We will prove first that this class of mechanisms always selects an assignment in the core.

Theorem 2 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism always selects an assignment in the core.*

As a corollary, we can deduce that all TTAS mechanisms satisfy Individual Rationality.

Corollary 3 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is individually rational.*

Now, we prove that TTAS maintains for the general case all the properties that characterize the TTC for the restricted case of strict preferences. We start with Pareto efficiency.

Theorem 4 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is Pareto-efficient.*

Additionally, we also prove that any mechanism of our family is strategy-proof.

Theorem 5 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is strategy-proof.*

Then, we have proved that, meanwhile in the restricted case of strict preferences the TTC mechanism is the only one that satisfies individual rationality, ex-post efficiency and strategy-proofness (see Ma [5]), in the general case we have a family of mechanisms that satisfy all these properties. Additionally, we are going to prove that our family of mechanisms always select an strict core allocation if the strict core is non-empty. That is, our family of mechanisms always generalizes the solution of Quint and Wako [6].

Theorem 6 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism selects an strict core allocation when the strict core is non-empty.*

4 Comments and Applications

We have proposed a family of deterministic mechanisms for the housing market problems for the general case in which individuals can report indifferences. This family of mechanisms generalizes the previous proposals of the TTC mechanism and the TTS algorithm, satisfying all the desirable properties that these proposals have. The mechanisms of the family differ in the priority ranking implemented to favor some individuals over others in case of conflict, but always without renouncing to the requirements of efficiency, strategy-proofness and the obligation of selecting a core allocation.

The selection of the priority ranking may be done in terms of some characteristic of the individuals that are not included in the formal specification of the housing market problem (income, seniority, ...). However, if there is not any intuitive way of selecting a priority ranking in a particular problem, it is always possible to randomize it. In this case, independently of the probability distribution over the priority rankings, we have that the random mechanism obtained satisfies individual rationality, strategy-proofness and ex-post efficiency[5]. Additionally, it selects a strict core allocation with probability 1 if the strict core is non-empty and, in general, it selects a core allocation with probability 1.

There are many other problems in the literature that can be seen as a exchange of indivisible goods: house allocation with existing tennants (Abdulkadiroglu and Sonmez [1]), kidney exchange (Roth et al. [9]), school choice (Abdulkadiroglu and Sonmez [2]). In all these problems, the proposed solution, maintaining the assumption that individuals can only report strict preferences, is based on adaptations of the TTC algorithm to these particular cases.[6] In these problems, it is also natural that individuals could have indifferences and, then, it is also necessary to propose a mechanism for the general case in which they can report them. We can easily adapt our family of Top Trading Absorbing Sets mechanisms to each of these problems in a similar way that the original TTC is adapted to incorporate the particular characteristics of each of these frameworks.[7] Therefore, the family of mechanisms that we propose has a wide range of problems in which they can be applied.

APPENDIX

We are going to prove all the results of the paper in the Appendix.

[5] This is a difference with respect to the mechanism of Yilmaz [11], which is ex-ante efficient.

[6] There is an exception in the school choice problem in which apart from the solution based in the TTC, Abdulkadiroglu and Sonmez [2]) also propose other mechanism based on the Gale-Shapley [3] deferred acceptance algorithm, which is the one that some authorities in US cities have selected to be applied.

[7] The particular details of the adaptations can be provided upon request.

Proof of proposition 1

By contradiction, suppose that this does not occur. That is, there is a maximal set of individuals $S \subseteq N$ and a maximal set of houses $T \subseteq H$ that are not removed never from the algorithm. Consider the algorithm just after the other agents and houses has been removed (suppose that this occurs in step i). Then, in this subgraph we know that there is at least one absorbing set. If at least one absorbing set is paired-symmetric, we have a contradiction. If, however, all absorbing sets are non paired-symmetric, we have that in each absorbing set A_i, there exist a set of nodes $B_i \subseteq A_i$ that do not belong to symmetric pairs. Then, we proceed with step (i.3) and we select one house for each individual to point using the priority ranking. Then, we go to step (i.4) and we proceed with the provisional trading of the houses in the cycles. The nodes of $A_i \setminus B_i$ will also belong to symmetric pairs in the next period. With respect to each node $\upsilon_i \in B_i$, if it is in a cycle, it will belong to a symmetric pair in the next period. Similarly, if there is no node of B_i in any cycle, it is easy to see that in the next period A_i is also an absorbing set.

Then, the set of nodes that belong to symmetric pairs are never decreasing. If they are increasing in some moments, we know that we will finally obtain a paired-symmetric absorbing set and, therefore, some agents and houses leave the algorithm and we arrive at a contradiction. Then, the unique possibility is that the set of agents B_i do not enter never in a cycle. However, we have seen that in this case the absorbing set A_i will stay stable over time. Given that any node of an absorbing set belongs to some cycle and that the selection of the house that each individual points to varies according to the rule of step (i.3), we can deduce that some node of B_i will finally enter in a selected cycle. Therefore, we have a contradiction and the proposition is proved.

Proof of Theorem 2

By contradiction, let μ be the assignment selected by applying the TTAS algorithm with priority ordering $\succ$ to some housing market problem (N, H, ω, R) and assume that μ is not in the core. Then, there exists a coalition $T \subseteq N$ and an assignment υ such that for all $i \in T$, $\upsilon_i \in \omega(T)$ and $\upsilon_i P_i \mu_i$. Denote, without loss of generality, $T = \{1, 2, \ldots, r\}$ such that $\upsilon_i = \omega_{i+1}$ for all $i \in \{1, \ldots, r-1\}$ and $\upsilon_r = \omega_1$. Take $1 \in T$. Given that $\upsilon_1 P_1 \mu_1$, we have that υ_1 have leaved the algorithm before 1. Then, $\omega^{-1}(\upsilon_1) = 2 \in T$ has entered in a cycle and has received a temporary assignment before 1. Moreover, 2 prefers υ_2 to μ_2 and this means that υ_2 have leaved the algorithm before 2. Then, $\omega^{-1}(\upsilon_2) = 3 \in T$ has entered in a cycle and has received a temporary assignment before 2. Following this argument, we have that for all $i \in \{1, \ldots, r-1\}$, $i+1$ has entered in a cycle and has received a temporary assignment before i and, therefore, r has entered in a cycle and has received a temporary assignment before 1. However, given that an individual and her initial endowment enters in a cycle by the first time in the same step, ω_1 is in the algorithm when r enters firstly in a cycle. Then, the house that r receives temporarily is not worse than $\upsilon_r = \omega_1$. Then, by Lemma 3, we can conclude that $\mu_r R_r \upsilon_r$, which is a contradiction. Then, the proposition is proved.

Proof of Theorem 4

By contradiction, suppose that there is some TTAS mechanism that selects for some housing market problem (N, H, ω, R) an assignment μ which is not Pareto efficient. That is, there exists an assignment ν such that for all $i \in N$, $\nu_i R_i \mu_i$ and for some $j \in N$, $\nu_j P_j \mu_j$. Given the construction of the algorithm, ν_j has leaved the algorithm with the agent $\mu^{-1}(\nu_j)$ before μ_j. This indicates that $\mu^{-1}(\nu_j)$ has belonged to a paired-symmetric absorbing set A in this moment. Given that $\mu^{-1}(\nu_j)$ should obtain by ν a house that is at least equally good from her than ν_j, we have that at least one agent z of A (probably $\mu^{-1}(\nu_j)$) have obtained by ν a house ν_z that have leaved the algorithm before z.

We can replicate the analysis with the agent $\mu^{-1}(\nu_z)$. This agent should have leaved the algorithm with a paired-symmetric absorbing set. Then, at least one agent w of this absorbing set have obtained by ν a house ν_w that have leaved the algorithm before w. However, this process can not be repeated infinitely: if we return continuously to symmetric absorbing sets that have leaved before, we will arrive at the first paired-symmetric absorbing set and it is impossible to return more. Therefore, we have a contradiction and the proposition is proved.

Proof of Theorem 5

We are going to prove some lemmas that will help us in the proof of the theorem. The first lemma states that all the houses that the TTAS algorithm assigns temporarily to an agent are indifferent to her.

Lemma 1 *Let x_i^t be the t-th temporary assignment that the TTAS algorithm assigns to agent i. Then $\forall t$ $x_i^t I_i x_i^{t+1}$.*

Proof. Consider the step of the algorithm in which x_i^{t+1} is assigned to agent i and let x_i^t be agent i's current assignment. Then, by construction of the algorithm, there is a cycle in which x_i^t points to i and i points to x_i^{t+1} and therefore x_i^{t+1} is maximal for i among the houses remaining in the market/algorithm. Hence $x_i^{t+1} R_i x_i^t$.

Now consider the step of the algorithm in which x_i^t was assigned to i. At this step, there was a cycle in which i points to x_i^t and ,by construction, x_i^t is maximal for i among the remaining houses. But in this step, house x_i^{t+1} is still in the market and therefore $x_i^t R_i x_i^{t+1}$.

Hence if $x_i^{t+1} R_i x_i^t$ and $x_i^t R_i x_i^{t+1}$ we can conclude that $x_i^t I_i x_i^{t+1}$ as desired. ■

Then, we can deduce the following corollary, by which the first house that the TTAS algorithm assigns temporarily to an agent determines the utility that this agent will have with her final assignment.

Corollary 7 *Let x and μ_i be the first temporary assignment and the final assignment that the TTAS algorithm assigns to agent i, respectively. Then $x I_i \mu_i$.*

The following lemma will also help us in the proof of the theorem. We will denote hereafter by $\varphi^{\succ}(P_{-i}, P_i)$ the TTAS mechanism with the priority ranking $\succ$ when the reported preferences are (P_{-i}, P_i) and the description of N, H and ω is clear.

Lemma 2 *Let h_k be the first house assigned temporarily to agent a_i by the TTAS algorithm for (P_{-i}, P_i) with priority ranking $\succ$ and let P_i' be any preference such that $\{h \in H \mid h P_i h_k\} = \{h \in H \mid h P_i' h_k\}$. Then,*

- *the set of cycles and paired-symmetric absorbing sets previous to the cycle assigning h_k to agent a_i in the algorithm defining $\varphi^{\succ}(P_{-i}, P_i)$ is also in the algorithm defining $\varphi^{\succ}(P_{-i}, P_i')$, and*
- *(ii) Each agent participates in the same sequence of temporal assignments in the algorithm defining $\varphi^{\succ}(P_{-i}, P_i)$ until agent a_i is assigned to h_k as in the first v stages of the algorithm defining $\varphi^{\succ}(P_{-i}, P_i')$.*

Proof. Consider that h_k and a_i enter in a selected cycle of the algorithm defining $\varphi^{\succ}(P_{-i}, P_i)$ in stage q.

Let $t = 1$ be the first step of the algorithm and let $G_1(P_{-i}, P_i)$ be the graph associated with this step when agent a_i declares P_i. Suppose that $q > 1$ (if not, the proof is finished). Notice that the paired-symmetric absorbing sets in $G_1^{\succ}(P_{-i}, P_i)$ (the digraph in step 1) are also in $G_1^{\succ}(P_{-i}, P_i')$. Let $\mathcal{C}_\infty$ denote the set of cycles obtained by the algorithm at the end of this step and let $c_j = \{a_1, h_2, a_2, h_3, ..., h_1\}$ be a cycle in $\mathcal{C}_\infty$. Now consider $G_1^{\succ}(P_{-i}, P_i')$. Notice that every agent in c_j is in this graph pointing to the same houses as in $G_1^{\succ}(P_{-i}, P_i)$ (given that $a_i \notin c_j$). (a) If c_j is in an absorbing set in $G_1^{\succ}(P_{-i}, P_i')$, then the same structure of priorities is used to select an arrow from each agent of c_j in $G_1^{\succ}(P_{-i}, P_i)$ and in $G_1^{\succ}(P_{-i}, P_i')$. Hence c_j is also obtained as a cycle in the first step of the algorithm for (P_{-1}, P_i'). (b) If not, all agents and houses in c_j will enter firstly in an absorbing set (the same absorbing set for all of them) in the same stage, of the algorithm. Then, in this stage, say stage t, the structure of priorities gives the same result as in $G_1^{\succ}(P_{-i}, P_i)$ and then the cycle c_j is also obtained in $G_t^{\succ}(P_{-i}, P_i')$ and this is the first cycle in which agents in c_i enter.

Consider now $t = 2$ (assume that $q > 2$, if not, the proof is finished) and let $G_2^{\succ}(P_{-i}, P_i)$ be the graph associated with this step when agent a_i declares P_i. Consider any paired-symmetric absorbing set, A_i, in this graph. It is easy to verify that every arrow from an agent in A_i to any house outside A_i in $G_1^{\succ}(P_{-i}, P_i)$ are not in $G_2^{\succ}(P_{-i}, P_i)$. This happens because these houses belong to a paired-symmetric absorbing set in $G_1^{\succ}(P_{-i}, P_i)$. Then, every arrow from an agent in A_i in $G_2^{\succ}(P_{-i}, P_i)$ are the same as in $G_2^{\succ}(P_{-i}, P_i')$ and in the subsequent stages (given that the paired-symmetric absorbing sets in $G_1^{\succ}(P_{-i}, P_i)$ and in $G_1^{\succ}(P_{-i}, P_i')$ are the same). Consider now a house $h_i \in A_i$. If h_i points to its original owner in A_i, it also points to her in $G_2^{\succ}(P_{-i}, P_i')$ and the subsequent stages. If h_i points to an agent a_j different from her original owner, it must belong to a cycle c_j obtained in $G_1^{\succ}(P_{-i}, P_i)$. Then, by the previous reasoning, we know that c_j will also be obtained in $G_t^{\succ}(P_{-i}, P_i')$ for some t. Therefore, we obtain that the paired-symmetric absorbing sets A_i will be obtained in $G_{t^*+1}^{\succ}(P_{-i}, P_i')$ (being t^* the later stage in which a house in A_i has entered in its corresponding cycle[8]). And, therefore, the sequence of temporal assignments that has received each of these agents are the same in both cases.

[8]If all of them point to their original owner, $t^* = 1$

Consider now a cycle $c_j = \{a_1, h_2, a_2, h_3, ..., h_1\}$ in $G_2^{\succ}(P_{-1}, P_i)$. In c_j there may be (only) three types of agents: (i) Those agents that have not entered in a selected cycle in the first stage in $G_1^{\succ}(P_{-i}, P_i)$ and point to some houses that are not present in $G_2^{\succ}(P_{-i}, P_i')$. (ii) Those agents that have not entered in a selected cycle in the first stage and do not belong to (i). (iii) Those agents that have entered in a selected cycle in the first stage. The houses that have disappeared in the first stage are the same in $G_1^{\succ}(P_{-i}, P_i)$ and in $G_1^{\succ}(P_{-i}, P_i')$ because the paired-symmetric absorbing sets are the same in both graphs (as we have proven above). Therefore, the agents in (i) point to the same houses in $G_2^{\succ}(P_{-i}, P_i)$ and in $G_2^{\succ}(P_{-i}, P_i')$ (and in the probably subsequent stages). By triviality, the agents in (ii) also point to the same houses in $G_2^{\succ}(P_{-i}, P_i)$ and in $G_2^{\succ}(P_{-i}, P_i')$. To respect to those agents in (iii), they point to all their maximal houses in $G_2^{\succ}(P_{-i}, P_i)$. We know that the cycle that each of them formed in the stage 1 of (P_{-i}, P_i) is also formed in some stage t of (P_{-i}, P_i). Let t^* be the later stage in which one of the cycles is formed ($t^* = 1$ if this set is empty). With respect to the houses of c_j, it is easy to verify that they point to the same agent in $G_2^{\succ}(P_{-i}, P_i)$ and in $G_{t^*+1}^{\succ}(P_{-i}, P_i')$. Then, $G_2^{\succ}(P_{-i}, P_i)|c_j = G_{t^*+1}^{\succ}(P_{-i}, P_i')|c_j$ [9]. Then we also have that there exists $\hat{t} \geq t^* + 1$ such that $G_2^{\succ}(P_{-i}, P_i)|c_j = G_{t^*+1}^{\succ}(P_{-i}, P_i')|c_j$ and c_j belongs to some absorbing set in both graphs. Given that c_j is formed in $G_2^{\succ}(P_{-i}, P_i)$, we can deduce that the house (h_j) that each agent (a_j) in c_j receives temporarily in this stage is the house with the highest priority between her maximal remaining houses. It is possible that in $G_{\hat{t}}^{\succ}(P_{-i}, P_i')$ the set of maximal remaining houses of each agent is a proper subset of that in $G_2^{\succ}(P_{-i}, P_i)$ but h^{j+1} is still present and, therefore, must be the house with the highest priority. Then c_i is also formed in stage $\hat{t}$ and the sequence of temporal assignments that has received each of these agents are the same in both cases.

When $t \in \{3, \ldots, q\}$, the proof is similar. Therefore, we have proved both parts of the lemma and v will correspond with the maximum of all $\hat{t}$ that will appear in the proof of all stages of the algorithm after a_i and h_k enter in a cycle. ■

[9]If G is a graph and c is a set of nodes of G, we denote by $G|c$ the restricted graph that includes only the nodes of c and the arrows that part from a node of c and arrive at a node of c.

Now, we prove that if when an individual attains a utility level declaring a preference, then there exist a house that gives this individual the same utility such that if the individual puts it as her maximal house, she will receive it by the algorithm.

Lemma 3 *Let $U_i(\varphi_i^{\succ}(P_{-i}, P_i)) = k$, then there exists a house h_j such that $U_i(h_j) = k$ and $\varphi_i^{\succ}(P_{-i}, P_i') = h_j$ for all P_i' with* $\max(R_i') = h_j$.

Proof. Let h_j be the first house assigned temporarily to agent a_i by the TTAS algorithm for (P_{-i}, P_i) when the priority ranking is $\succ$. Notice that by Corollary 7, $U_i(h_j) = U_i(\varphi_i^{\succ}(P_{-i}, P_i) = k$. By Lemma 2, we have that the absorbing set of agent a_i in the graph corresponding to the stage q of the algorithm in which h_j is assigned to a_i when she declares P_i is the same as the absorbing set of agent a_i in the graph corresponding to the stage v of the algorithm when she declares P_i'. Additionally, we know that each agent has passed from the same sequence of temporal assignments in both algorithms until these steps. Then, if the priority criterion has selected the cycle in which h_j is assingmed a_i when she declares P_i, the priority criterion has to select also this cycle when she declares P_i'. Given that $\max(R_i') = h_j$, we have by Corollary 7 that $\varphi_i^{\succ}(P_{-i}, P_i') = h_j$. ■

Now, we can prove the theorem. By Lemma 3, we have that if there is any way of obtaining a particular level of utility, it is also possible to obtain this level declaring any preference in which the maximal house is one of the houses (h_k) that gives you this utility. Consider in particular a ranking P_i' in which h_k is the unique maximal house, $h_k P_i \varphi_i(P_{-i}, P_i)$ and $\{h \in H \,|\, hP_ih_k\} = \{h \in H \,|\, hP_i'h_k\}$. Then, by Lemma 2, we have that the set of cycles and paired-symmetric absorbing sets that have been formed when she declares P_i before obtaining $\varphi_i^{\succ}(P_{-i}, P_i)$ are also formed when she declares P_i'. Then, in particular, we know that h_k has belonged to a paired-symmetric absorbing set and has leaved the algorithm with an agent different from a_i when a_i declares P_i. Therefore, h_k has also leaved the algorithm with this different agent when a_i declares P_i'. Then, it is impossible for a_i to obtain a better house than $\varphi_i^{\succ}(P_{-i}, P_i)$ and the theorem is proved.

Proof of Theorem 6

Consider a housing market problem (N, H, ω, R) with a non-empty strict core. We need to introduce an algorithm, called Top Trading Segmentation (hereafter, TTS) originally proposed by Quint and Wako (2004) to determine a partition of the set of agents and houses.

Step 1: Let each agent point to her maximal houses and each house point to its owner. Select the absorbing sets of this digraph. Each absorbing set constitutes an element of the partition.

Step i: Let each agent point to her maximal houses among the remaining ones and each remaining house point to its owner. Select the absorbing sets of this digraph. Each absorbing set constitutes an element of the partition.

With the partition obtained with the algorithm, Quint and Wako (2004) proved that the following statements are equivalent:

- In each element of the partition, it is possible to find a sub-allocation that assigns to each agent one of their maximal houses in this set.
- The strict core of the problem is non-empty and one of the allocations of it consists of the union of all these sub-allocations.

Now, we will prove the following lemma.

Lemma 4 *Consider a digraph (V, E) such that V is an absorbing set. Then, we can partition V in a set of disjoint cycles if and only if there exists a subset $E' \subseteq E$ such that in the digraph (V, E') all nodes have indegree and outdegree equal to 1.*

Proof. Assume first that we can partition V in a set of disjoint cycles, $\{c^i = (h^i_1, a^i_1, h^i_2, a^i_2, \ldots, h^i_{m_i}, a^i_{m_i})\}_{i \in \{1,\ldots,k\}}$. Then, construct E' in the following way: $(x, y) \in E'$ if and only if $[x = a^i_j$ and $y = h^i_{j+1}]$ or $[x = h^i_j$ and $y = a^i_j]$ for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$. Then, it is easy to see that in (V, E') all nodes have indegree and outdegree equal to 1.

Assume now that there exists a subset $E' \subseteq E$ such that in the digraph (V, E') all nodes have indegree and outdegree equal to 1. Then, start with any node of the digraph as the first node of a cycle. Then, continue with the unique sequence of edges (given that the outdegree of all nodes is 1) that part from this node. Given that the indegree of all nodes is equal to 1, this sequence will terminate in the initial node in some moment. Then, this sequence is the first cycle of the partition. Starting with other node that does not belong to this cycle, we will construct other cycle, disjoint from the first one. Finally, following this procedure, we will have a partition of V in a set of disjoint cycles. ■

Then, consider any absorbing set of the first step of the TTAS algorithm. Note that this absorbing set is also one of the sets of the partition that TTS determines. Then, given the result of Quint and Wako (2004), we have that in a housing market problem with a non-empty strict core, the absorbing sets determined in step 1 of the TTAS algorithm must have a partition in disjoint cycles. Or, equivalently, using Lemma 4, in each of these absorbing sets we can find a subset of edges such that the indegree and outdegree of each of these nodes is equal to 1.

If the absorbing set is paired-symmetric, our algorithm gives to each agent in step (1.2) one of her maximal houses. Given Corollary 7, we deduce that our mechanism allocates to these individuals one of their maximal houses.

If the absorbing set is non paired-symmetric, our algorithm applies in step (1.3) a priority ranking $\succ$ to determine only one edge for each node of the absorbing set, and in step (1.4), the resulting cycles trade provisionaly their houses. If the priority ranking chooses exactly the edges that determine the partition in cycles of the strict core allocation, we will have that all individuals of the absorbing set will attain provisionally one of their maximal houses. Given Corollary 7, we deduce that our mechanism allocates to these individuals one of their maximal houses.

If, however, the priority ranking chooses other different edges, we need to prove that in the second step of the algorithm we have that the same agents belong to a partition of absorbing sets such that we can find a subset of edges such that the indegree and outdegree of each of these nodes is equal to 1. Given that this condition is satisfied in the step 1, we are going to construct a function that assigns, for each of the edges belonging to this subset in step 1, an edge in the step 2. First, for each edge from an agent to a house, consider exactly the same edge in step 2. Second, for the edges from houses to agents, select the unique edge that part from each house in step 2. Now, we will prove that taking into account these edges, the nodes will have indegree and outdegree equal to 1. Given that we have selected only one edge starting from any node, the outdegree is equal to 1 in all nodes. To see why the indegree is also equal to 1, consider first the nodes of the agents. In this nodes, only the house that in this moment belong to this individual points to, and, then, the indegree is equal to 1. Consider now the nodes of the houses. Given that we have selected the same edges that in step 1 of the algorithm and the condition was satisfied then, we have that the indegree is here also equal to 1 for any node.

Finally, if the condition is satisfied for all the absorbing sets determined in step 1 of the TTAS algorithm, it is easy to see that the other sets of the partition obtained by the TTS algorithm will appear in subsequent steps of the TTAS algorithm and the same reasoning applies. Therefore, the theorem is proved.

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